

Seria Fourier. Analiza spectrală a semnalelor periodice

Daca descompunem semnalul de intrare periodic intr-o serie de componente mai simple, putem calcula raspunsul la fiecare componenta si face sinteza raspunsurilor partiale.

In domeniul frecventa: seria Fourier.

<http://shannon.etc.upt.ro/teaching/ssist/Cap4.pdf>

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Răspunsul sistemelor continue liniare și invariante în timp la exponențiala complexă de modul unitar

$$x(t) = e^{j\omega_0 t} \xrightarrow{\quad} \boxed{h(t)} \xrightarrow{\quad} y(t) = \int_{-\infty}^{\infty} h(\tau) e^{j\omega_0(t-\tau)} d\tau$$

$$y(t) = e^{j\omega_0 t} \cdot \underbrace{\int_{-\infty}^{\infty} h(\tau) e^{-j\omega_0 \tau} d\tau}$$

$H(\omega_0)$. Transformata Fourier a raspunsului la impuls h , calculata in ω_0 : depinde de ω_0 si h

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$$x(t) = e^{j\omega_0 t} \xrightarrow{\text{Funcție proprie a SLIT}} \boxed{h(t)} \xrightarrow{\text{Valoare proprie a SLIT}} y(t) = e^{j\omega_0 t} \cdot H(\omega_0)$$

$$H(\omega) = \int_{-\infty}^{\infty} h(\tau) e^{-j\omega\tau} d\tau = |H(\omega)| e^{j\Phi(\omega)}$$

$$y(t) = e^{j\omega_0 t} H(\omega_0) = |H(\omega_0)| e^{j(\omega_0 t + \Phi(\omega_0))}$$

$$x(t) = \sum_k a_k e^{j\omega_k t} \xrightarrow{\boxed{h(t)}} y(t) = \sum_k a_k H(\omega_k) e^{j\omega_k t}$$

Daca semnalul de intrare este o combinatie liniara de exponentiale complexe \Rightarrow iesirea : o combinatie liniara de exponentiale complexe

$$y(t) = \sum_k a_k \mathcal{S}\{e^{j\omega_k t}\} = \sum_k a_k H(\omega_k) e^{j\omega_k t}$$

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Transformari ortogonale

- Produsul scalar al vectorilor

$$\mathbf{x} = [x_1 \ x_2 \ \dots \ x_n]^T ; \quad \mathbf{y} = [y_1 \ y_2 \ \dots \ y_n]^T$$

$$\langle \mathbf{x}, \mathbf{y} \rangle = [x_1 \ x_2 \ \dots \ x_n] \begin{bmatrix} y_1^* \\ y_2^* \\ \dots \\ y_n^* \end{bmatrix} = x_1 y_1^* + x_2 y_2^* + \dots + x_n y_n^*$$

- Produsul scalar al functiilor din $L^2_{[a,b]}$

$$\langle x(t), y(t) \rangle = \int_a^b x(t) y^*(t) dt$$

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- Se observa ca se indeplinesc urmatoarele conditii:

i) $\langle x, y \rangle = \langle y, x \rangle^*$,

ii) $\langle x, y + z \rangle = \langle x, y \rangle + \langle x, z \rangle$,

iii) $\langle \lambda x, y \rangle = \lambda \langle x, y \rangle$,

iv) $\langle x, \lambda y \rangle = \lambda^* \langle x, y \rangle \quad \forall \lambda \in \mathbb{C}$,

v) $\left\langle \sum_{k=1}^n \alpha_k x_k, \sum_{l=1}^m \beta_l y_l \right\rangle = \sum_{k=1}^n \sum_{l=1}^m \alpha_k \beta_l^* \langle x_k, y_l \rangle$.

- Norma $\|x\|$ este finita (spatiul L^2):

$$\|\mathbf{x}\|^2 = \langle \mathbf{x}, \mathbf{x} \rangle = |x_1|^2 + |x_2|^2 + \dots + |x_n|^2 = \sum_{k=1}^n |x_k|^2$$

$$\|x\|^2 = \int_a^b |x(t)|^2 dt$$

Un spatiu vectorial cu norma definita prin produsul scalar este un spatiu Hilbert (teoria aproximarii)

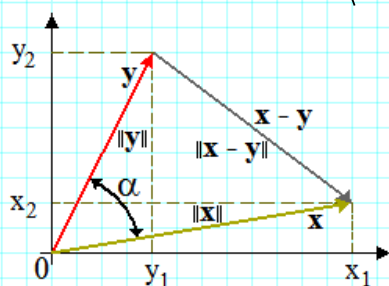
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Vectori ortogonali (perpendicularari)

- Pentru doi vectori bidimensionali

$$\mathbf{x} = \mathbf{i}x_1 + \mathbf{j}x_2 ; \mathbf{y} = \mathbf{i}y_1 + \mathbf{j}y_2$$

- Produsul scalar : $\langle \mathbf{x}, \mathbf{y} \rangle = \|\mathbf{x}\| \|\mathbf{y}\| \cos \alpha$



$$\cos \alpha = \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{\|\mathbf{x}\| \cdot \|\mathbf{y}\|}$$

α -unghiul dintre vectori

Conditia de ortogonalitate : Produsul scalar sa fie zero

$$\langle \mathbf{x}, \mathbf{y} \rangle = 0 \Leftrightarrow \mathbf{x} \perp \mathbf{y}$$

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Functii ortogonale

- Vom considera doua semnale definite pe $(0, T_0)$, cu $T_0 = 2\pi/\omega_0$ – **spatiul $L^2_{[0, T_0]}$**

$$x(t) = \cos \omega_0 t ; y(t) = \sin \omega_0 t$$

- Produsul scalar este:

$$\begin{aligned} \langle \cos \omega_0 t, \sin \omega_0 t \rangle &= \int_0^{T_0} \cos(\omega_0 t) \sin(\omega_0 t) dt = \frac{1}{2} \int_0^{T_0} \sin(2\omega_0 t) dt \\ &= -\frac{\cos(2\omega_0 t)}{4\omega_0} \Big|_0^{T_0} = \frac{1 - \cos 4\pi}{4\omega_0} = 0 \end{aligned}$$

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Spatiul Hilbert

- Un sistem $U = \{u_k\}$ de vectori ortogonali doi cate doi se spune ca este complet in spatiul Hilbert, H , daca nu exista nici un vector $x \in H - U$, care sa fie ortogonal pe toti vectorii din U (doar vectorul $\mathbf{0}$):

$$\langle u_k, x \rangle = 0 \Leftrightarrow x = \mathbf{0}, \text{ if } x \in H - U.$$

- Un sistem complet U formeaza baza ortogonala in spatiul Hilbert. Pentru orice element x din H , exista o **dezvoltare unica** de forma

$$\forall x \in H, x = \sum_k a_k u_k.$$

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Exemple

- Daca multimea elementelor din U , n , este finita: spatiu Hilbert finit dimensional, cu dimensiunea n .
- Daca multimea este numarabila dar infinita: spatiu Hilbert infinit dimensional.
- Versorii $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ formeaza o baza in spatiul tridimensional, cu $n=3$.
- Multimea functiilor $\{e^{jk\omega_0 t}\}_{k \in \mathbb{Z}}$ cu frecventa $k\omega_0$ – o baza infinit dimensionala pentru semnale periodice in timp continuu, de perioada T_0

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Teorema lui Pitagora in spatiul Hilbert.

Relatia dintre distanta si produsul scalar

- Fie diferenta intre doi vectori din spatiul Hilbert

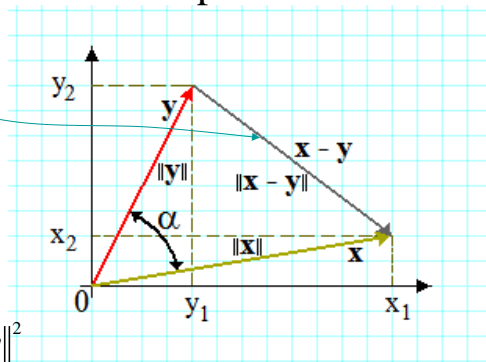
$$\mathbf{d} = \mathbf{x} - \mathbf{y}$$

- “Distanta” dintre ei:

$$d^2(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|^2$$

- Avem in general:

$$\begin{aligned} d^2(x, y) &= \|x - y\|^2 \\ &= \|x\|^2 - 2\operatorname{Re}\{\langle x, y \rangle\} + \|y\|^2 \end{aligned}$$



- Daca x si y sunt ortogonali $d^2(x, y) = \|x\|^2 + \|y\|^2$

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Exemple, $L^2_{[0,T_0]}$

- Norma pentru semnalele ortogonale $\cos(\omega_0 t)$ si $\sin(\omega_0 t)$

$$\|x(t)\|^2 = \int_0^{T_0} \cos^2(\omega_0 t) dt = \int_0^{T_0} \frac{1 + \cos(2\omega_0 t)}{2} dt = \frac{t}{2} \Big|_0^{T_0} + \frac{1}{2} \cdot \frac{1}{2\omega_0} \sin(2\omega_0 t) \Big|_0^{T_0} = \frac{T_0}{2}$$

- Distanța dintre semnale este (cf. teoremei lui Pitagora)

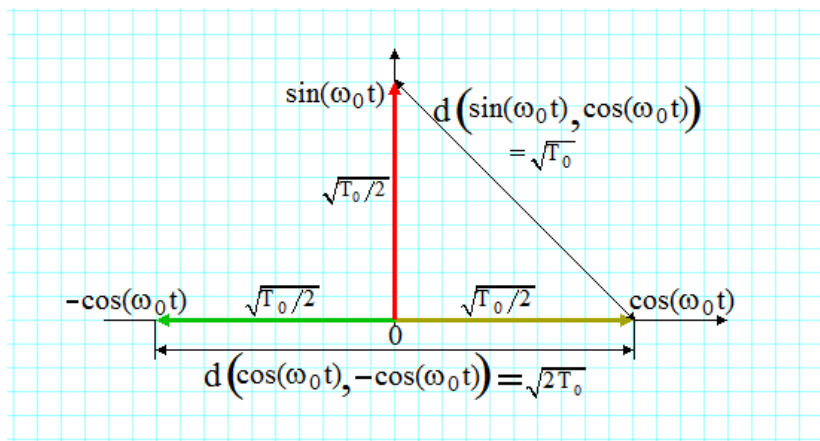
$$d^2(\cos \omega_0 t, \sin \omega_0 t) = \frac{T_0}{2} + \frac{T_0}{2} = T_0.$$

- Semnalele ce nu sunt ortogonale nu satisfac teorema lui Pitagora. Ex: $\cos(\omega_0 t)$ si $-\cos(\omega_0 t)$

$$\langle \cos \omega_0 t, -\cos \omega_0 t \rangle = -\int_0^{T_0} \cos^2 \omega_0 t dt = -T_0/2.$$

$$d^2(\cos \omega_0 t, -\cos \omega_0 t) = T_0/2 + T_0/2 - 2(-T_0/2) = 2T_0.$$

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$$d(\cos \omega_0 t, \sin \omega_0 t) < d(\cos \omega_0 t, -\cos \omega_0 t).$$

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Inegalitatea lui Schwartz in spatiul Hilbert

(Cauchy- Bunyakovsky-Schwarz)

$$|\langle x, y \rangle| \leq \|x\| \cdot \|y\|$$

Egalitatea are loc daca si numai daca x si y sunt dependente liniar, $y=kx$, unde $k=const.$

- Semnale ortogonale $L^2_{[0, T_0]}$

$$x(t) = \cos(\omega_0 t), y(t) = \sin(\omega_0 t) \Rightarrow |\langle x(t), y(t) \rangle| = 0$$

- Produsul normelor este:

$$\|x(t)\| \cdot \|y(t)\| = \sqrt{\frac{T_0}{2}} \cdot \sqrt{\frac{T_0}{2}} = \frac{T_0}{2}$$

- Inegalitate $0 < T_0/2$. Nu exista o const. k pt care $y=kx$.

- Semnale ne-ortogonale $L^2_{[0, T_0]}$

$$x(t) = \cos(\omega_0 t); y(t) = -\cos(\omega_0 t) \Rightarrow y(t) = -x(t)$$

- Egalitate. Exista o const. $k=-1$ pentru care $y=kx$.

$$\|x(t)\| = \sqrt{\frac{T_0}{2}}; \|y(t)\| = \sqrt{\frac{T_0}{2}}; |\langle x(t), y(t) \rangle| = \left| -\frac{T_0}{2} \right| = \frac{T_0}{2}$$

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Cea mai buna aproximare a unui vector in spatiul Hilbert

- Consideram un spatiu Hilbert, H , n -dimensional, $U = \{u_1, \dots, u_n\}$ in care este definita o baza ortonormala,

$$\langle u_k, u_l \rangle = \begin{cases} \|u_l\|^2, & k=l \\ 0, & k \neq l \end{cases}; \|u_k\|^2 = 1$$

- Coeficientii sunt calculabili cu $c_k = \langle x, u_k \rangle$.
- **Cea mai buna aproximare:** reprezentarea m -dimensionala, retinand m termeni din cei n :

$$\tilde{x} = \sum_{k=1}^m c_k u_k$$

- Cu cat creste numarul de termeni din aproximare, m , descreste eroarea si aproximarea devine mai buna

- **Trunchierea seriei.**

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Teorema proiectiei

- Fie H -spatiu Hilbert, H_s –subspatiu Hilbert. Oricare ar fi vectorul x din H , exista un vector \tilde{x} din H_s care este **cea mai buna aproximare** a sa
 1. Distanța de la x la \tilde{x} , este mai mica decât distanța de la x la oricare alt vector din H_s
 2. Eroarea de aproximare $e = x - \tilde{x}$ este ortogonală pe subspatiul H_s

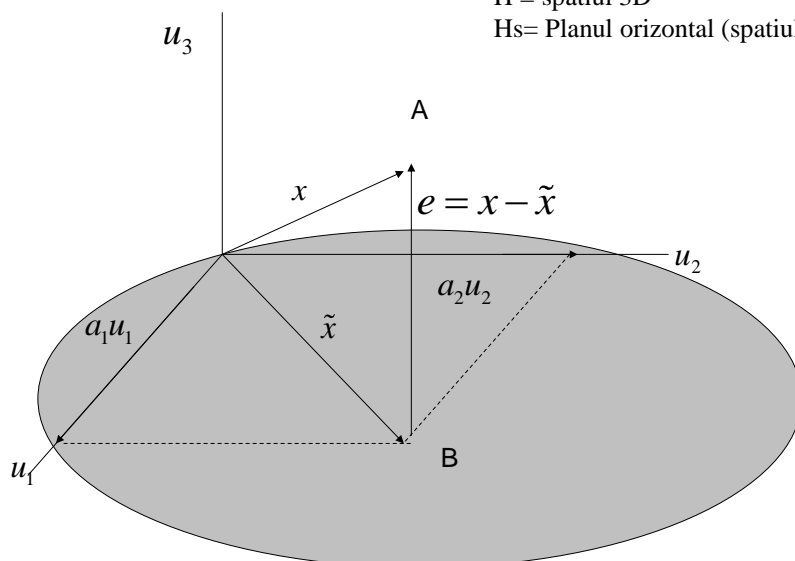
$$d_{\min}^2(x, \tilde{x}) = \|x\|^2 - \|\tilde{x}\|^2$$

$$\|x - \tilde{x}\|^2 = \|x\|^2 - \|\tilde{x}\|^2 \Rightarrow \|x\|^2 = \|\tilde{x}\|^2 + \|x - \tilde{x}\|^2$$

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$$x = \overline{OA}, \tilde{x} = \overline{OB}, e = \overline{BA}$$

$H = \text{spatiul 3D}$
 $H_s = \text{Planul orizontal (spatiul 2D)}$



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Spatiul Hilbert infinit dimensional

- $U_N = \{u_k(t), k = -N, N\}$ baza ortogonală finită.

Descompunerea semnalului se face:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k u_k(t), \text{ with } c_k = \frac{\langle x(t), u_k(t) \rangle}{\|u_k(t)\|^2}$$

- **Aproximarea se face tot prin trunchiere:**

$$\tilde{x}_N(t) = \sum_{k=-N}^N c_k u_k(t)$$

- cu eroare minima

$$\|x(t) - \tilde{x}_N(t)\|^2 = \|x(t)\|^2 - \sum_{k=-N}^N |c_k|^2 \|u_k(t)\|^2$$

- Cu cat mai multi termeni (N mare): eroarea scade

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$$\|x(t)\|^2 = \sum_{k=-\infty}^{\infty} |c_k|^2 \|u_k(t)\|^2 \quad \text{Relatia lui Parseval}$$

- Eroarea devine :

$$\|x(t) - \tilde{x}_N(t)\|^2 = \sum_{k=-\infty}^{\infty} |c_k|^2 \|u_k(t)\|^2 - \sum_{k=-N}^N |c_k|^2 \|u_k(t)\|^2 = \sum_{\forall |k| > N} |c_k|^2 \|u_k(t)\|^2$$

- **Inegalitatea lui Bessel**

$$\|\tilde{x}_N(t)\|^2 = \sum_{k=-N}^N |c_k|^2 \|u_k(t)\|^2 \leq \|x(t)\|^2$$

$$\|x(t)\|^2 < \infty \text{ fiindca } x(t) \in L^2_{[a,b]}$$

$$\Rightarrow \lim_{N \rightarrow \infty} \sum_{\forall |k| > N} |c_k|^2 \|u_k(t)\|^2 = 0 \Rightarrow \lim_{N \rightarrow \infty} \|x(t) - \tilde{x}_N(t)\|^2 = 0$$

- Semnalul de aproximare $\tilde{x}_N(t)$ converge in **medie patratica** catre $x(t)$

$$\text{l.i.m.}_{N \rightarrow \infty} \tilde{x}_N(t) = x(t)$$

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Remarci

1. Avem $\|x(t)\|^2 = \|\tilde{x}_N(t)\|^2 + \|x(t) - \tilde{x}_N(t)\|^2$

Teorema lui Pitagora: ortogonalitate intre cea mai buna aproximare si eroarea de aproximare

$$\langle \tilde{x}_N(t), x(t) - \tilde{x}_N(t) \rangle = 0$$

2. Relatia lui **Parseval** (teorema energiei, **Rayleigh**)

$$W = \|x(t)\|^2 = \sum_{k=-\infty}^{\infty} |c_k|^2 \|u_k(t)\|^2$$

3. Cea mai buna aproximare se obtine prin trunchierea seriei

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Seria Fourier exponentiala

• In spatiul $L^2_{[0, T_0]}$ consideram baza ortogonala:

$$u_k(t) = e^{jk\omega_0 t}, \quad k \in \mathbb{Z}$$

$$\langle e^{jk\omega_0 t}, e^{jl\omega_0 t} \rangle = \int_0^{T_0} e^{j(k-l)\omega_0 t} dt = \begin{cases} 0, & k \neq l \\ T_0, & k = l \end{cases}; \text{Norma } \|u_k(t)\|^2 = T_0$$

• Pentru un semnal periodic $x(t) = x(t + T_0)$

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \leftrightarrow c_k = \frac{1}{T_0} \int_{T_0} x(t) e^{-jk\omega_0 t} dt, \quad \omega_0 = \frac{2\pi}{T_0}$$

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Seria Fourier trigonometrica

- Relatiile lui Euler

$$\cos(k\omega_0 t) = \frac{1}{2}(e^{jk\omega_0 t} + e^{-jk\omega_0 t}); \quad \sin(k\omega_0 t) = \frac{1}{2j}(e^{jk\omega_0 t} - e^{-jk\omega_0 t})$$

- O baza ortogonală :

$$U = \{1, \cos(k\omega_0 t), \sin(k\omega_0 t)\}_{k \in \mathbb{N}}$$

$$\|1\|^2 = T_0; \quad \|\cos(k\omega_0 t)\|^2 = \|\sin(k\omega_0 t)\|^2 = \frac{T_0}{2}$$

- Orice semnal periodic, de perioada T_0 poate fi exprimat sub forma

$$x(t) = a_0 \cdot 1 + \sum_{k=1}^{\infty} (a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t))$$

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Seria Fourier trigonometrica

- Coeficientii seriei sunt:

$$a_0 = \frac{\langle x(t), 1 \rangle}{\|1\|^2} = \frac{1}{T_0} \int_{T_0} x(t) dt, \quad \text{componenta continua}$$

$$a_k = \frac{\langle x(t), \cos(k\omega_0 t) \rangle}{\|\cos(k\omega_0 t)\|^2} = \frac{2}{T_0} \int_{T_0} x(t) \cos(k\omega_0 t) dt,$$

$$b_k = \frac{\langle x(t), \sin(k\omega_0 t) \rangle}{\|\sin(k\omega_0 t)\|^2} = \frac{2}{T_0} \int_{T_0} x(t) \sin(k\omega_0 t) dt.$$

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Cateva observatii

1. a_0 - componenta continua DC a semnalului $x(t)$
2. Semnalul fara componenta continua ($a_0=0$) are numai componente "oscilante" :

$$x(t) = \sum_{k=1}^{\infty} (a_k \cos k\omega_0 t + b_k \sin k\omega_0 t);$$

$$x(t) - \text{impar} \Rightarrow a_k = 0; \quad x(t) - \text{par} \Rightarrow b_k = 0;$$

3. Pentru semnale reale:

$$x(t) = x^*(t) \Rightarrow c_{-k} = c_k^*$$

$$c_{-k} = \frac{1}{T_0} \int_{T_0} x(t) e^{jk\omega_0 t} dt = \left[\frac{1}{T_0} \int_{T_0} x(t) e^{-jk\omega_0 t} dt \right]^* = c_k^*$$

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4. Puterea semnalului $x(t)$ – relatia lui Parseval:

$$P = \frac{W}{T_0} = \sum_{k=-\infty}^{\infty} |c_k|^2 \frac{T_0}{T_0} \Rightarrow P = \sum_{k=-\infty}^{\infty} |c_k|^2 = \frac{1}{T_0} \int_{T_0} |x(t)|^2 dt$$

- O alta forma:

$$P = \frac{1}{T_0} \int_{T_0} |x(t)|^2 dt = a_0^2 + \sum_{k=1}^{\infty} \left(\frac{a_k^2}{2} + \frac{b_k^2}{2} \right)$$

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Seria Fourier armonica

- Folosind relatia:

$$a_k \cos k\omega_0 t + b_k \sin k\omega_0 t = \sqrt{a_k^2 + b_k^2} \cos(k\omega_0 t + \varphi_k)$$

$$\operatorname{tg} \varphi_k = -\frac{b_k}{a_k}. A_k = \sqrt{a_k^2 + b_k^2}$$

- Seria Fourier trigonometrica devine:

$$x(t) = \sum_{k=0}^{\infty} A_k \cos(k\omega_0 t + \varphi_k)$$

- **Forma armonica.**

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Relatii intre coeficienti

- Pentru semnale reale

$$|c_k| = \sqrt{a_k^2 + b_k^2} = \frac{1}{2} A_k, k \geq 1$$

$$|c_k| = |c_{-k}|, k \leq -1;$$

$$\arg c_k = \varphi_k, k \geq 1;$$

$$\arg c_{-k} = -\varphi_k, k \leq -1;$$

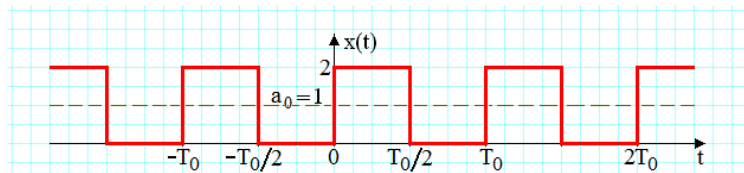
$$|c_0| = |a_0|; \arg c_0 = 0.$$

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Diagrame spectrale pentru semnalele reale

- Semnalele periodice se pot reprezenta in domeniul frecventa.

$$x(t) = \begin{cases} 2, & 0 \leq t < \frac{T_0}{2} \\ 0, & \frac{T_0}{2} \leq t < T_0 \end{cases}$$



Semnal rectangular, factor de umplere (duty cycle) 0.5

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- Componenta continua DC:

$$a_0 = \frac{1}{T_0} \int_{T_0} x(t) dt = \frac{1}{T_0} \int_0^{T_0/2} 2 dt = 1; \quad A_0 = a_0$$

- Partea oscilanta este impara

$$x_1(t) = \begin{cases} 1, & 0 \leq t < \frac{T_0}{2} \\ -1, & \frac{T_0}{2} \leq t < T_0 \end{cases} \quad k \neq 0 \Rightarrow a_k = 0$$

$$b_k = \frac{2}{T_0} \int_{T_0} x(t) \sin k\omega_0 t dt = \frac{4}{T_0} \cdot \frac{-\cos k\omega_0 t}{k\omega_0} \Big|_0^{T_0/2} = \frac{4}{T_0} \cdot \frac{1 - (-1)^k}{k\omega_0}; \quad k \geq 1$$

$$b_{2k-1} = \frac{4}{(2k-1)\pi}; \quad k = 1, 2, 3, \dots \quad b_{2k} = 0$$

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$$x(t) = 1 + \sum_{k=1}^{\infty} \frac{4}{(2k-1)\pi} \sin[(2k-1)\omega_0 t]$$

$$x(t) = 1 + \sum_{k=1}^{\infty} \underbrace{\frac{4}{(2k-1)\pi}}_{A_{2k-1}} \cos\left[(2k-1)\omega_0 t - \frac{\pi}{2}\right]$$

armonica de ordinul $(2k-1)$, frecventa $(2k-1)\omega_0$

- Forma armonica

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Diagrama spectrala de amplitudini ($k\omega_0, A_k$)

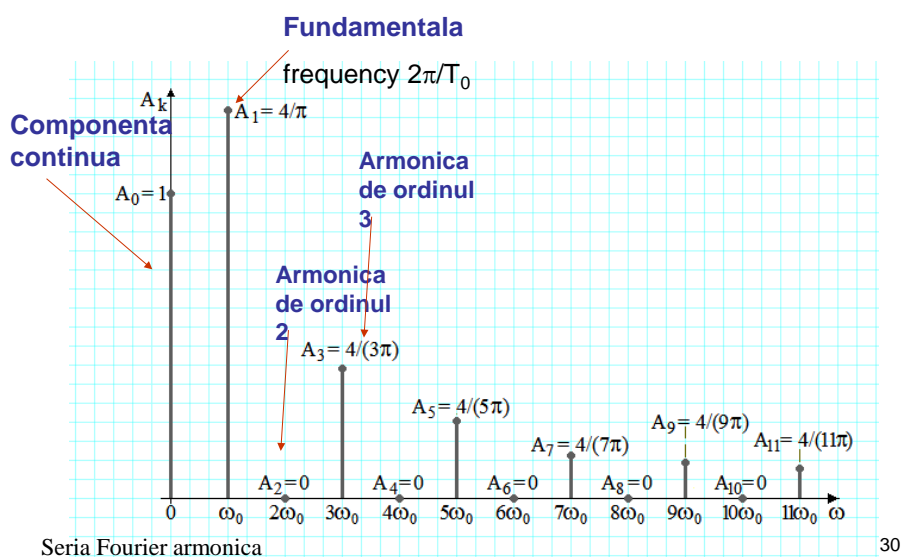
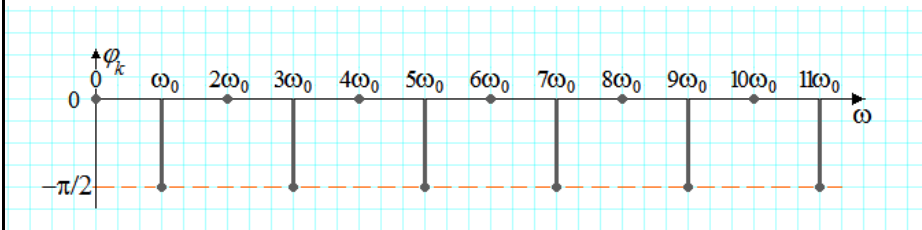


Diagrama spectrala de faze ($k\omega_0, \varphi_k$)



Seria Fourier armonica

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Diagrama spectrala de modul ($k\omega_0, |c_k|$)

- Se porneste de la seria Fourier **exponentiala**
- Coeficientii c_k sunt:

$$c_0 = \frac{1}{T_0} \int_{T_0} x(t) dt = a_0 = 1$$

$$c_k = \frac{1}{T_0} \int_{T_0} x(t) e^{-jk\omega_0 t} dt = \frac{1}{T_0} \int_0^{T_0/2} 2e^{-jk\omega_0 t} dt = \frac{1 - (-1)^k}{jk\pi}; \quad k \neq 0$$

$$c_{2k-1} = \frac{2}{|2k-1|\pi} e^{j\frac{\pi}{2}}; \quad k \leq -1; \quad c_{2k-1} = \frac{2}{(2k-1)\pi} e^{-j\frac{\pi}{2}}; \quad k \geq 1$$

$$c_{2k} = 0, \quad k \neq 0$$

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Diagrama spectrala de modul ($k\omega_0, |c_k|$)

$$|c_{2k-1}| = \frac{2}{|2k-1|\pi}$$

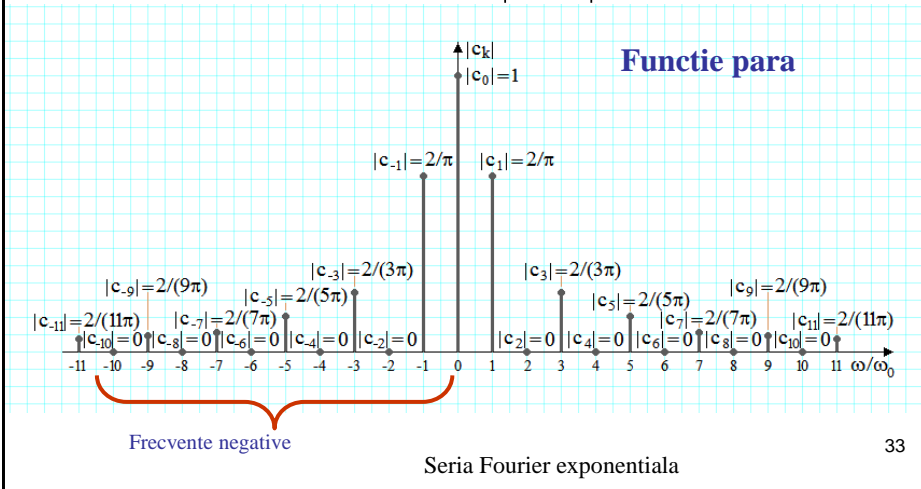
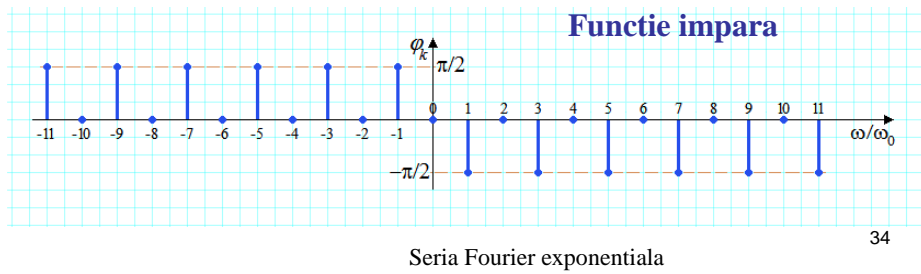


Diagrama spectrala de faze, pentru $\omega > 0$ și $\omega < 0$, ($k\omega_0, \varphi_k$)

$$\varphi_k = -\frac{\pi}{2} \operatorname{sgn}(k)$$



Alte forme ale relatiei lui Parseval

- Seria Fourier exponentiala :

$$P = \frac{1}{T_0} \int_{T_0} |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |c_k|^2 = |c_0|^2 + 2 \sum_{k=0}^{\infty} |c_k|^2$$

- Forma trigonometrica si armonica

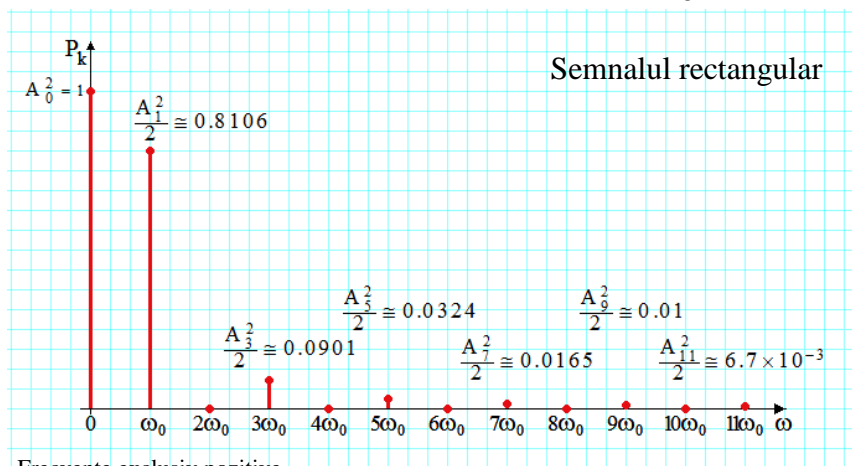
$$P = a_0^2 + \sum_{k=1}^{\infty} \left(\frac{a_k^2}{2} + \frac{b_k^2}{2} \right) = \frac{1}{T_0} \int_{T_0} |x(t)|^2 dt = A_0^2 + \sum_{k=1}^{\infty} \frac{A_k^2}{2}$$

- Exemflu. Puterea semnalului rectangular:

$$P = \frac{1}{T_0} \int_{T_0} |x(t)|^2 dt = \frac{1}{4} \int_0^{T_0/2} 4 dt = 2$$

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Diagrama spectrala de putere folosind seria Fourier armonica ($k\omega_0$, $A_k^2/2$)

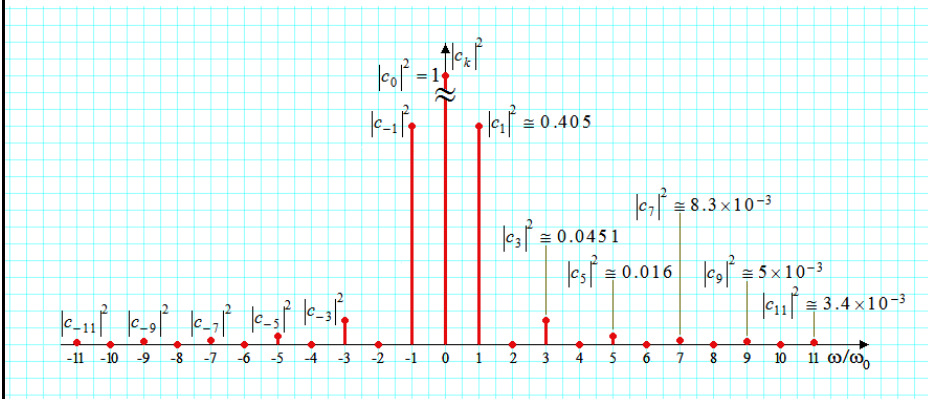


Frecvente exclusiv pozitive

Se recomanda folosirea unei reprezentari logaritmice pentru putere ce avantajeaza reprezentarea puterilor mai putin semnificative

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Diagrama spectrala de putere folosind seria Fourier exponentiala ($k\omega_0, |c_k|^2$)



Frecvente pozitive si negative

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- Pentru semnale de banda nelimitata :
 - Banda de frecventa este infinita.
 - Puterea scade cu cresterea frecventei, tinde spre zero pt frecvente ce tind la infinit
- ***Banda efectiva de frecvente*** = gama pozitiva de frecvente ce contin un procentaj semnificativ al puterii semnalului.
- In acest caz, in banda $9\omega_0$ se gaseste **96,5%** din puterea semnalului.

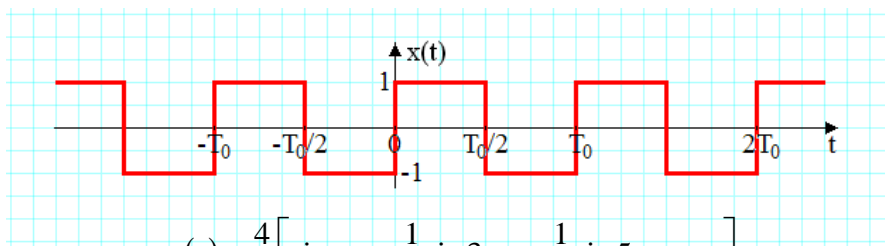
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Fenomenul Gibbs

- Fizicianul **Albert Michelson** a construit un analizor de spectru in 1898.
- La iesirea filtrelor analizorului, a masurat amplitudinile componentelor spectrale, conform teoriei
- Cand a incercat sa recompuna prin insumare, semnalul initial, a observat ca apare o “problema”, la semnal anume. L-a rugat pe **Gibbs** sa ii explice acest fenomen.

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Semnalul analizat (de banda nelimitata): rectangular cu factor de umplere 0.5, fara componenta continua



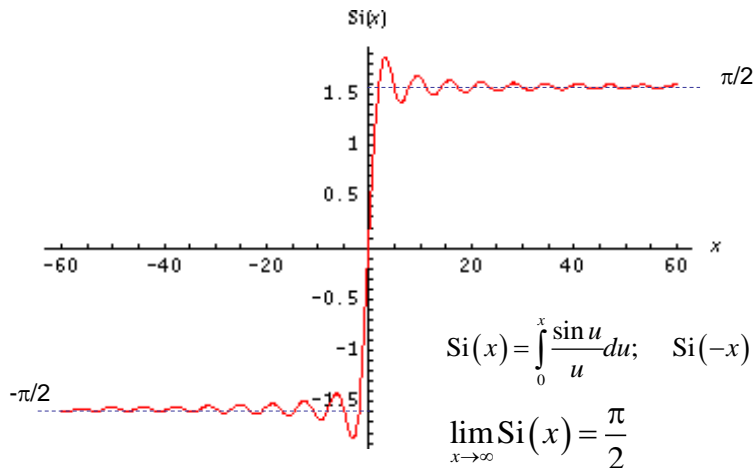
$$x(t) = \frac{4}{\pi} \left[\sin \omega_0 t + \frac{1}{3} \sin 3\omega_0 t + \frac{1}{5} \sin 5\omega_0 t + \dots \right]$$

Printr-o trunchiere in frecventa, pastrand primele n armonici, de ordin impar, semnalul este aproximat cu unul de banda limitata:

$$\tilde{x}(t) = \frac{4}{\pi} \left[\sin \omega_0 t + \frac{1}{3} \sin 3\omega_0 t + \frac{1}{5} \sin 5\omega_0 t + \dots + \frac{1}{2n-1} \sin (2n-1)\omega_0 t \right] \quad 40$$

$$\tilde{x}(t) = \frac{2}{\pi} \int_0^{2n\omega_0 t} \frac{\sin u}{u} du = \frac{2}{\pi} \text{Si}(2n\omega_0 t)$$

- Si(x) – sinus integral, functie impară
<http://mathworld.wolfram.com/SineIntegral.html>



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Fenomenul Gibbs

- Gibbs a aratat ca trunchiind semnalul rectangular cu factor de umplere 0.5, si pastrand n armonici de ordin impar,

$$\tilde{x}(t) = \frac{4}{\pi} \left[\sin \omega_0 t + \frac{1}{3} \sin 3\omega_0 t + \frac{1}{5} \sin 5\omega_0 t + \dots + \frac{1}{2n-1} \sin (2n-1)\omega_0 t \right]$$

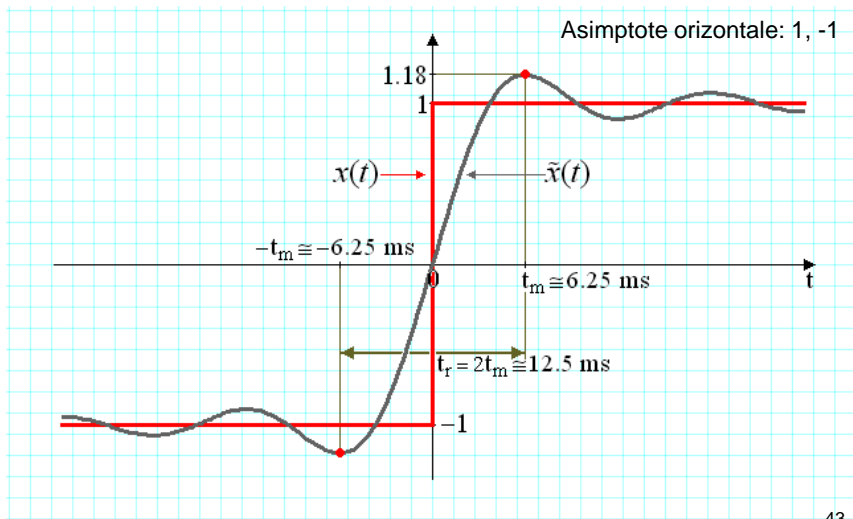
- Se obtine

$$\tilde{x}(t) = \frac{2}{\pi} \int_0^{2n\omega_0 t} \frac{\sin u}{u} du = \frac{2}{\pi} \text{Si}(2n\omega_0 t)$$

- Semnalele de banda nelimitata nu pot fi perfect aproximata cu semnale de banda limitata.

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O unda rectangulara cu $T_0=1\text{s}$, cu $2nf_0=80f_0$



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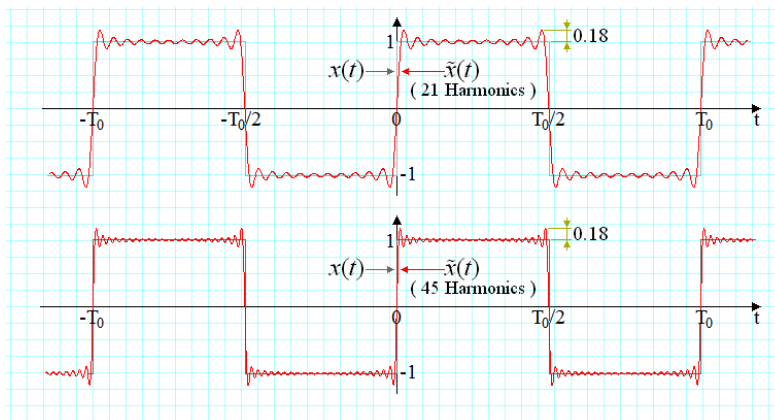
- Prima supracrestere (maximul oscilatiei), de 1.18 V apare la momentul $t_m=6,25\text{ms} \ll T_0$
- Oscilatii in vecinatatea zonei de tranzitie.
- Depasirea valorii 1 este de 9%
- Timpul de ridicare (rise time)

$$t_r \cong 2t_m = \frac{2\pi}{\omega_M} = \frac{1}{f_M}$$

$\omega_M=(2n-1)\omega_0$ aproximativ $2n\omega_0$ – cea mai mare frecventa retinuta

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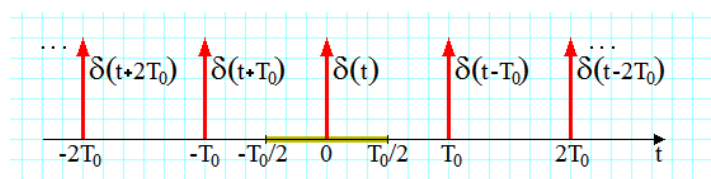
Semnale trunchiate pentru 21 si respectiv 45 armonici



Se poate observa ca oscilatiile nu scad ca si amplitudine, dar frecventa lor creste. Semnalul de aproximare converge in medie patratica catre semnalul $x(t)$.

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Distributia Dirac periodica



$$\delta_T(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT) \longrightarrow c_k = \frac{1}{T_0}$$

$$c_k = \frac{1}{T} \int_{-T/2}^{T/2} \delta_T(t) e^{-jk \frac{2\pi}{T} t} dt = \frac{1}{T} \int_{-T/2}^{T/2} \delta(t) dt = \frac{1}{T}$$

- Pentru $[-T/2, T/2]$, $\delta_T(t) = \delta(t)$.

$$\delta_T(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT) = \frac{1}{T} \sum_{k=-\infty}^{\infty} e^{jk\omega_0 t}$$

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Proprietatile seriei Fourier exponentiale

- Coeficientii seriei Fourier a semnalului x , de perioada T

$$x(t) \xleftrightarrow{\mathcal{F}} \{c_k^x\}$$

$$c_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt$$

- Descompunerea Fourier

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \quad \text{a.p.t. (aproape peste tot)}$$

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1. Liniaritatea

- semnalele $x(t)$ si $y(t)$ periodice cu perioada T :

$$x(t) \xleftrightarrow{\mathcal{F}} \{c_k^x\} \quad , \quad y(t) \xleftrightarrow{\mathcal{F}} \{c_k^y\}$$

$$ax(t) + by(t) \xleftrightarrow{\mathcal{F}} \{ac_k^x + bc_k^y\}$$

2. Deplasarea în timp

$$x(t - t_0) \xleftrightarrow{\mathcal{F}} \{e^{-jk\omega_0 t_0} c_k^x\}$$

$$c'_k = \frac{1}{T} \int_T x(t - t_0) e^{-jk\omega_0 t} dt = \frac{1}{T} \int_T x(\tau) e^{-jk\omega_0(\tau + t_0)} d\tau = e^{-jk\omega_0 t_0} c_k^x$$

- Deplasarea in timp \rightarrow modulatie cu exponentiala complexa

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3. Conjugarea complexă

- Conjugarea complexa in timp → reflectarea in domeniul frecventa si conjugarea complexa

$$x^*(t) \leftrightarrow (c_{-k}^x)^*$$

$$c_k' = \frac{1}{T_0} \int_{T_0} x^*(t) e^{-jk\omega_0 t} dt = \left[\frac{1}{T_0} \int_{T_0} x(t) e^{-j(-k)\omega_0 t} dt \right]^* = (c_{-k}^x)^*$$

4. Reflectarea semnalului

$$c_k' = \frac{1}{T} \int_T x(-t) e^{-jk\omega_0 t} dt = \frac{1}{T} \int_T x(\tau) e^{-j(-k)\omega_0 \tau} d\tau = c_{-k}^x$$

$$\tilde{x}(t) = x(-t) \xleftrightarrow{\mathcal{F}} \{c_{-k}^x\}$$

- Reflectarea in timp → reflectare in frecventa

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5. Scalarea variabilei timp

- $x(t)$ - perioada $T \Rightarrow x(at)$, perioada $T/|a|$.

$$c_k' = \frac{1}{T/a} \int x(at) e^{-jk\omega_0 t} dt; \quad \omega_0' = \frac{2\pi}{T/a} = a\omega_0$$

$$c_k' = \frac{1}{T} \int x(\tau) e^{-jk\omega_0 \tau} d\tau = c_k^x$$

$$x(at) \xleftrightarrow{\mathcal{F}} \{c_k^x\}$$

6. Modularea semnalului

- Modulatia in timp → deplasare in domeniul frecventa

$$c_k' = \frac{1}{T} \int x(t) e^{jk_0\omega_0 t} e^{-jk\omega_0 t} dt = \frac{1}{T} \int x(t) e^{-j(k-k_0)\omega_0 t} dt = c_{k-k_0}^x$$

$$x(t) e^{jk_0\omega_0 t} \xleftrightarrow{\mathcal{F}} \{c_{k-k_0}^x\}$$

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Dualitatea timp-frecventa

- O operatie in timp \rightarrow alta operatie in frecventa:
 - De exemplu: **modulatie in timp** \rightarrow **deplasare in frecventa**
- A doua operatie in timp \rightarrow prima operatie in frecventa.
 - **Deplasare in timp** \rightarrow **modulatie in frecventa**
- Acest comportament este numit dualitate.
- Reflectarea este o operatie auto-duala

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7. Produsul a două semnale

- Convolutia coeficientilor.

$$x(t)y(t) \xleftrightarrow{\mathcal{F}} \left\{ \sum_{n=-\infty}^{\infty} c_{k-n}^x c_n^y \right\} = \{c_k^x * c_k^y\}$$

8. Convoluția periodică a semnalelor

- Semnalele periodice nu au energie finita, si convolutia nu se poate defini. Se foloseste convolutia circulara sau periodica, definita pe o perioada.

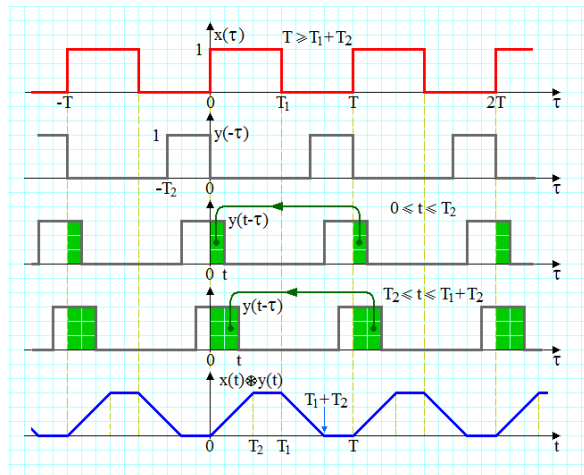
$$z(t) = \int_T x(\tau) y(t-\tau) d\tau = x(t) \circledast y(t) \xleftrightarrow{\mathcal{F}} \{T c_k^x c_k^y\}$$

- Operatii duale: inmultirea \leftrightarrow convolutia

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Convoluția periodică a doua semnale rectangulare, cu factor de umplere diferit

- Efect de circularitate.



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9. Derivarea semnalului

- După derivare, componenta continuă = 0. Semnalul rămâne periodic. Derivarea în timp → înmulțirea spectrului cu $jk\omega_0$.

$$\frac{dx(t)}{dt} \xleftrightarrow{\mathcal{F}} \{jk\omega_0 c_k^x\}$$

10. Integrarea semnalului

- Pentru ca semnalul să rămână periodic după integrare, componenta continuă trebuie să fie nulă. Integrarea în timp → înmulțirea spectrului cu $1/jk\omega_0$.

$$\int_{-\infty}^t x(\tau) d\tau \xleftrightarrow{\mathcal{F}} \left\{ \frac{c_k^x}{jk\omega_0} \right\} \quad c_0^x = 0$$

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11. Semnale reale. Seriile componentelor para si impara

- $x(t)$ semnal real; $c_k = c_{-k}^*$
- Componentele para $x_p(t)$ si impara $x_i(t)$.
- Spectrul componentei pare $x_p(t)$ –real
- Spectrul componentei impare (semnal real) $x_i(t)$ – pur imaginar

$$x_p(t) = \frac{x(t) + x(-t)}{2} \xleftrightarrow{\mathcal{F}} \{\operatorname{Re} c_k^x\}$$

$$x_i(t) = \frac{x(t) - x(-t)}{2} \xleftrightarrow{\mathcal{F}} \{j \operatorname{Im} c_k^x\}$$

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