Fourier transform of continuous-time signals

Spectral representation of non-periodic signals

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Fourier transform: aperiodic signals

- repetition of a **finite-duration signal x(t)=> periodic signals**.

$$\tilde{x}(t) = x(t) * \delta_T(t) = x(t) * \sum_{k=-\infty}^{\infty} \delta(t - kT) = \sum_{k=-\infty}^{\infty} x(t - kT)$$

- Periodic signal (**T** $\rightarrow \infty$) => non-periodic signal x(t)

$$\tilde{x}(t) \overset{T \rightarrow \infty}{\rightarrow} x(t)$$

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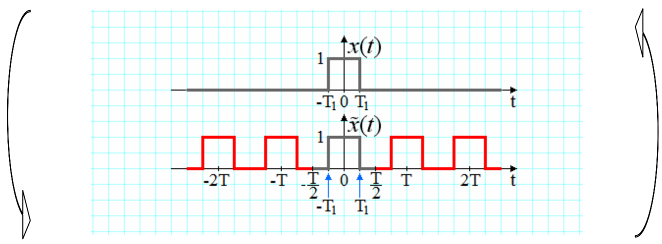
Non-periodic signal & periodic signal, period T.

$$x(t) \overset{\Delta}{=} p_{T_1}(t) = \begin{cases} 1, & |t| \le T_1 \\ 0, & \text{otherwise} \end{cases}$$

Non-periodic

repetition



$T \rightarrow \infty$

Periodic

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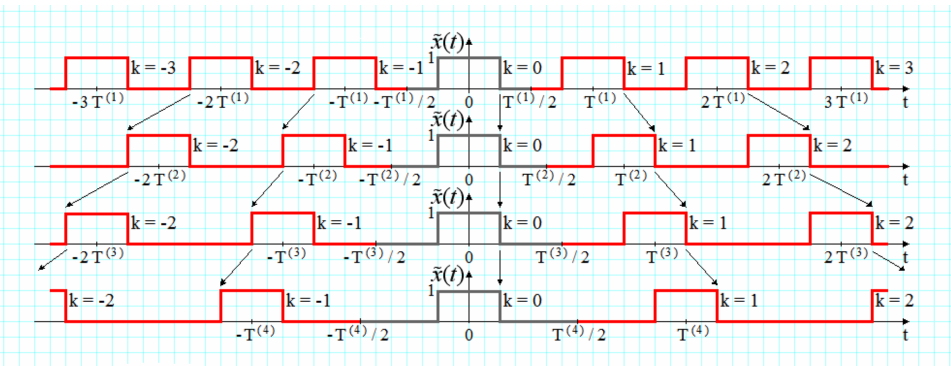

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} x(t-kT) \qquad \tilde{x}(t) \overset{T\to\infty}{\rightarrow} x(t)$$

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- analyze a non-periodic signal in the frequency domain:
  - using the frequency analysis of the correspondent periodic signal
  - and compute the limit for $T \to \infty$.

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$$c_k = \frac{2 \sin k\omega_0 T_1}{k\omega_0 T} \quad k \neq 0$$

- The periodic signal is non band-limited.

$$Tc_k = \begin{cases} \dfrac{2 \sin k\omega_0 T_1}{k\omega_0} ; & k \neq 0 \\ 2T_1 ; & k = 0 \end{cases}$$

$$X(\omega) = \frac{2 \sin \omega T_1}{\omega}$$

$$Tc_k = \begin{cases} X(k\omega_0) ; & k \neq 0 \\ 2T_1 ; & k = 0 \end{cases}$$

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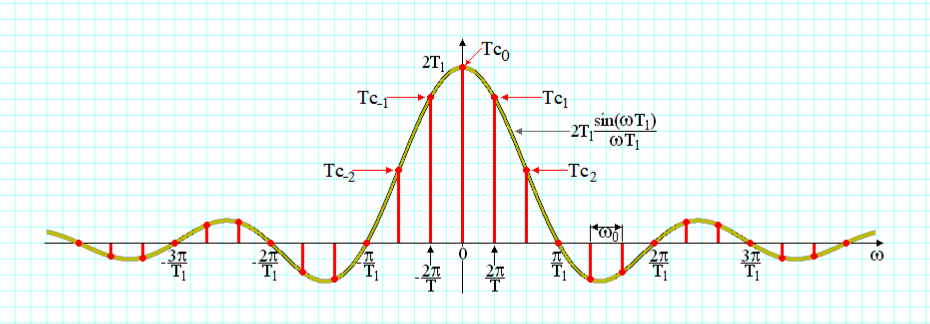
The product $T \cdot c_k$ & the envelope

$X(\omega)$ = envelope for $T \cdot c_k$



Relation between them?

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General case

- the Fourier coefficients of the periodic signal :

$$c_k = \frac{1}{T} \int_{-T/2}^{T/2} \tilde{x}(t) e^{-jk\omega_0 t} dt$$

Equal on [-T/2, T/2]

$$c_k = \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-jk\omega_0 t} dt$$

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Fourier transform

- outside [-T/2,T/2] the non-periodic signal =0

$$c_k = \frac{1}{T}\int_{-\infty}^{\infty} x(t)e^{-jk\omega_0 t}dt$$

- With the function:

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

(The envelope of $T\cdot c_k$)

$$c_k = \frac{1}{T}X(k\omega_0),\ \ \omega_0 = \frac{2\pi}{T}$$

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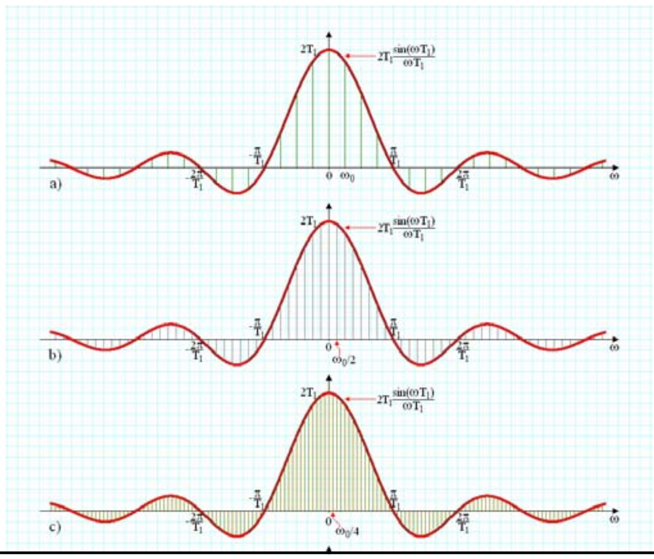
Square wave:different values of *T*



$$\omega_0 = \frac{2\pi}{T}$$

$$X(\omega) = \frac{2\sin\omega T_1}{\omega}$$

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Remarks

- The envelope is not affected by *T*.
- Increase T $\Rightarrow$ spectral components are "closer".
- $T \to \infty \Rightarrow$
  - distance$\to$0
  - the discrete spectral representation becomes continuous.
  - the periodic signal $\to$ non-periodic.

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Definition

- Fourier pair

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega \leftrightarrow X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

Inverse Fourier Transform

Fourier Transform (spectrum)

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Remarks

- periodic signals : spectral lines

$$c_k = \frac{1}{T} X(k\omega_0),\ \omega_0 = \frac{2\pi}{T}$$

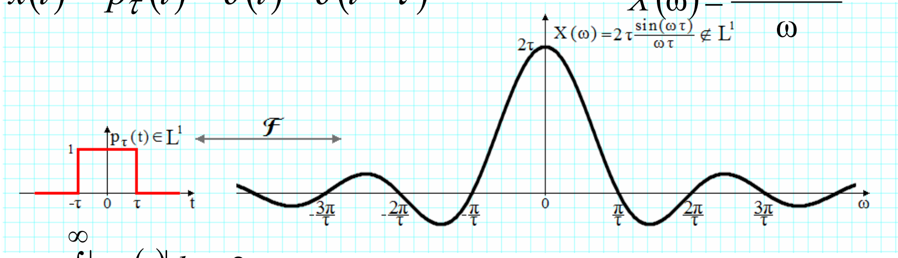
- non-periodic signals spectra are continuous

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

CTFT for signals in the class L^1

- for signals in L^1, the Fourier transform is not necessarily from L^1
- Reconstruction theorem!

$$x(t) = p_\tau(t) = \sigma(t) - \sigma(t-\tau) \qquad X(\omega) = \frac{2\sin\omega\tau}{\omega}$$



$$\int_{-\infty}^{\infty} |p_\tau(t)| dt = 2 < \infty$$

- the Fourier transform is convergent (the signal $x(t) \in L^1$) but $X(\omega) \notin L^1$.
- the reconstruction of the signal from its spectrum is not obvious.

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Reconstruction theorem

- If the signal $x(t)$ belongs to L^1 and has bounded variation on the entire real axis then its Fourier transform can be inverted using :

$$x(t) = \lim_{R \to \infty} \int_{-R}^{R} \mathcal{F}^{1}\{x(t)\}(\omega) \cdot e^{j\omega t} d\omega$$

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1. Linearity

- If $x(t)$ and $y(t) \in L^1$ and have the Fourier transform $X(\omega)$ and $Y(\omega)$ then for any complex constants a and b the signal $ax(t)+by(t) \in L^1$ and has the Fourier transform $aX(\omega)+bY(\omega)$.

$$\boxed{ax(t)+by(t) \rightarrow aX(\omega)+bY(\omega)}$$

Homework: Prove it.

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2. Time Shifting

- Time shifting -> modulation in frequency (multiplication with a complex exponential).

$$\boxed{x(t-t_0) \leftrightarrow e^{-j\omega t_0} X(\omega)}$$

- Proof

$$\mathcal{F}^1\{x(t-t_0)\} = \int_{-\infty}^{\infty} x(t-t_0) \cdot e^{-j\omega t} dt \overset{\tau=t-t_0}{=} \int_{-\infty}^{\infty} x(\tau) \cdot e^{-j\omega(\tau+t_0)} d\tau = e^{-j\omega t_0} X(\omega).$$

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Remarks

- Fourier transform: complex function.
- The Fourier transform H(ω) of the impulse response h(t) of a system: **frequency response**.
- frequency dependence of the magnitude of H(ω) = **magnitude characteristic** of the system |H(ω)|
- frequency dependence of the argument of H(ω) = **phase characteristic** of the system arg{H(ω)}

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3. Modulation

- Modulation in time -> shifting in frequency.

$$e^{j\omega_0 t}x(t) \rightarrow X(\omega-\omega_0).$$

- Proof

$$\mathcal{F}^1\left\{x(t)\cdot e^{j\omega_0 t}\right\} = \int_{-\infty}^{\infty} x(t)\cdot e^{j\omega_0 t}\cdot e^{-j\omega t}dt =$$

$$= \int_{-\infty}^{\infty} x(t)\cdot e^{-j(\omega-\omega_0)t}dt = X(\omega-\omega_0)$$

$$e^{j\omega_0 t}x(t) \rightarrow X(\omega-\omega_0).$$

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Duality

- operation in time $\Rightarrow$ another operation in frequency :
  - modulation $\Rightarrow$ shifting (3rd property)
- 2nd operation in time $\Rightarrow$ first operation in frequency.
  - time shifting $\Rightarrow$ modulation (2nd property)
- This behavior is named **duality**.

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4. Time Scaling

- If $x(t) \in L^1 \Rightarrow$ its scaled version $x(t/a) \in L^1$ and the spectrum of $x(t/a)$ is a frequency scaled version of the spectrum of $x(t)$.
- the scaling is an auto-dual operation.

$$x(at) \rightarrow \frac{1}{|a|} X\left(\frac{\omega}{a}\right).$$

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- Proof

$$\mathcal{F}^1\{x(at)\} = \int_{-\infty}^{\infty} x(at)\cdot e^{-j\omega t}dt \stackrel{\tau=at}{=} \frac{1}{|a|}\int_{-\infty}^{\infty} x(\tau)\cdot e^{-j\frac{\omega}{a}\tau}d\tau = \frac{1}{|a|}X\left(\frac{\omega}{a}\right);$$

$$x(at)\rightarrow \frac{1}{|a|}X\left(\frac{\omega}{a}\right).$$

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Example: the square wave

- spectrum

$$p_\tau(t)\leftrightarrow 2\frac{sin\,\omega\tau}{\omega}$$

- its time-scaled variant , a=2:

$$p_\tau(2t) = p_{\frac{\tau}{2}}(t)\leftrightarrow \frac{2}{2}\frac{sin\frac{\omega}{2}\tau}{\frac{\omega}{2}} = \frac{2\,sin\frac{\omega}{2}\tau}{\omega}$$

- a=1/2

$$p_\tau\left(\frac{1}{2}t\right) = p_{2\tau}(t)\leftrightarrow \frac{2}{\frac{1}{2}}\frac{sin\,2\omega\tau}{2\omega} = 2\frac{sin\,2\omega\tau}{\omega}$$

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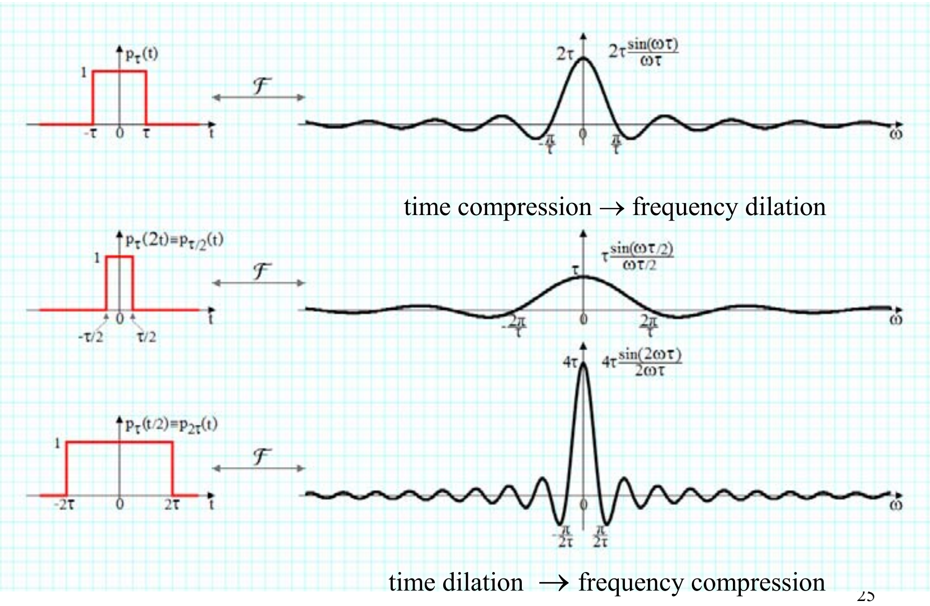

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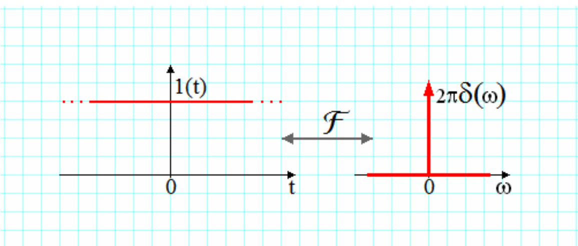
CTFT of the constant distribution

$$1(t) \overset{F}{\leftrightarrow} 2\pi\delta(\omega)$$



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Proof

- the constant distribution can be approximated:

$$\lim_{\tau\to\infty} p_\tau(t) = 1(t)$$

- We know that

$$\int_{-\infty}^{\infty} p_\tau(t)\cdot e^{-j\omega t}dt = 2\frac{\sin\omega\tau}{\omega}$$

$$\lim_{\tau\to\infty}\int_{-\infty}^{\infty} p_\tau(t)\cdot e^{-j\omega t}dt = \lim_{\tau\to\infty} 2\frac{\sin\omega\tau}{\omega}$$

$$\int_{-\infty}^{\infty} 1(t)\cdot e^{-j\omega t}dt = \lim_{\tau\to\infty} 2\tau\frac{\sin\omega\tau}{\omega\tau} = \begin{cases}\infty, & \omega = 0\\ 0, & \omega \neq 0\end{cases}$$

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- The area under the graphical representation of the spectrum:

$$A = \int_{-\infty}^{\infty} 2\tau\frac{\sin\omega\tau}{\omega\tau}d\omega = 2\left(\int_{-\infty}^{0}\frac{\sin u}{u}du + \int_{0}^{\infty}\frac{\sin u}{u}du\right) = 4Si(\infty) = 2\pi$$

- So: $$\int_{-\infty}^{\infty} 1(t)\cdot e^{-j\omega t}dt = \begin{cases}\infty, & \omega = 0\\ 0, & \omega \neq 0\end{cases}$$

- and: $$A = 2\pi$$

$$\int_{-\infty}^{\infty} 1(t)\cdot e^{-j\omega t}dt = 2\pi\delta(\omega) \Leftrightarrow 1(t) \overset{\mathcal{F}}{\leftrightarrow} 2\pi\delta(\omega)$$

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- An immediate consequence: a new representative string for the Dirac distribution:

$$\lim_{\tau \to \infty} \frac{\sin \omega\tau}{\pi\omega} = \delta(\omega)$$

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5. Complex Conjugation

- complex conjugation in time -> reversal and complex conjugation in frequency.

$$x^*(t) \overset{\mathcal{F}^1}{\leftrightarrow} X^*(-\omega)$$

- Proof

$$\mathcal{F}^1\{x^*(t)\} = \int_{-\infty}^{\infty} x^*(t) \cdot e^{-j\omega t} dt = \left[\int_{-\infty}^{\infty} x(t) \cdot e^{-j(-\omega)t} dt \right]^* = X^*(-\omega)$$

$$x^*(t) \overset{\mathcal{F}^1}{\leftrightarrow} X^*(-\omega)$$

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6. Time Reversal

- Time reversal -> reversal in frequency.
- Homework. Prove it.

$$x(-t) \overset{\mathcal{F}^1}{\leftrightarrow} X(-\omega)$$

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7. Signal's Derivation

- Time differentiation -> multiplication with $j\omega$ in frequency.

$$x'(t) \overset{\mathcal{F}^1}{\leftrightarrow} j\omega \cdot X(\omega)$$

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Proof: $$\mathcal{F}^1\{x'(t)\} = \int_{-\infty}^{\infty} x'(t)\cdot e^{-j\omega t}dt$$

Integrating by parts:

$$\mathcal{F}^1\{x'(t)\} = x(t)\cdot e^{-j\omega t}\Big|_{-\infty}^{\infty} + j\omega\int_{-\infty}^{\infty} x(t)\cdot e^{-j\omega t}dt$$

the signal is in L[1] :

$$\lim_{t\to\pm\infty}\left|x(t)\cdot e^{-j\omega t}\right| = \lim_{t\to\pm\infty}|x(t)| = 0$$

So: $$x'(t)\overset{\mathcal{F}^1}{\leftrightarrow} j\omega\cdot X(\omega)$$

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8. Signal's Integration

- For $x(t) \in L^1$ with $X(0)=0$ (**no DC component**), its integral $\in L^1$
- Time integration -> multiplication with

  $1/j\omega$ in frequency

$$\int_{-\infty}^{t} x(\tau)d\tau \overset{F^1}{\leftrightarrow} \frac{X(\omega)}{j\omega} \text{ for } X(0)=0$$

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Proof

- We have: $y(t)=\int_{-\infty}^{t} x(\tau)d\tau$
- Apply for $y(t)$ the differentiation property:

$$y'(t)=x(t)\overset{\mathcal{F}^1}{\leftrightarrow} j\omega Y(\omega)=X(\omega)\Rightarrow Y(\omega)=\frac{X(\omega)}{j\omega}$$

- Y defined in 0 :

$$X(0)=0$$

- So: $$\int_{-\infty}^{t} x(\tau)d\tau \overset{\mathcal{F}^1}{\leftrightarrow} \frac{X(\omega)}{j\omega}$$

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9. Signals' convolution
convolution theorem

- the convolution of two signals from L^1 belongs to L^1.
- convolution of two signals in time -> product in frequency.

$$x(t)*y(t)\overset{\mathcal{F}^1}{\leftrightarrow} X(\omega)\cdot Y(\omega)$$

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Proof:

$$\mathcal{F}^1\{x(t)*y(t)\}=\int_{-\infty}^{\infty}(x*y)(t)\cdot e^{-j\omega t}dt=\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}x(\tau)y(t-\tau)d\tau\right]\cdot e^{-j\omega t}dt=$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}x(\tau)\cdot e^{-j\omega\tau}\cdot x(t-\tau)\cdot e^{-j\omega(t-\tau)}dtd\tau\overset{t-\tau=u}{=}\int_{-\infty}^{\infty}x(\tau)\cdot e^{-j\omega\tau}d\tau\int_{-\infty}^{\infty}y(u)\cdot e^{-j\omega u}du.$$

$$x(t)*y(t)\overset{\mathcal{F}^1}{\leftrightarrow}X(\omega)\cdot Y(\omega)$$

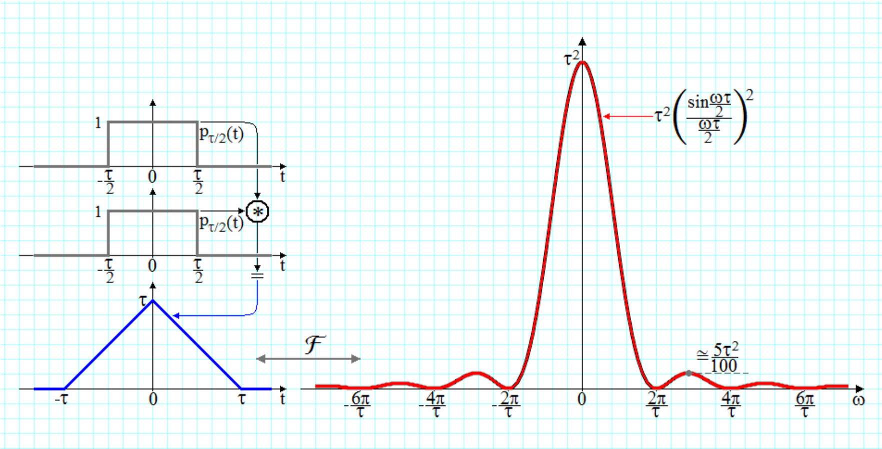
Example. Triangle's spectrum

convolution of two rectangular pulses, same duration= a triangle

$$p_{\frac{\tau}{2}}(t)*p_{\frac{\tau}{2}}(t)=\tau\left(1-\frac{|t|}{\tau}\right)p_{\tau}(t)$$

$$p_{\frac{\tau}{2}}(t)\leftrightarrow 2\frac{sin\frac{\omega\tau}{2}}{\omega}=\tau\frac{sin\frac{\omega\tau}{2}}{\frac{\omega\tau}{2}}$$

$$\tau\left(1-\frac{|t|}{\tau}\right)p_{\tau}(t)\leftrightarrow\tau^2\left(\frac{sin\frac{\omega\tau}{2}}{\frac{\omega\tau}{2}}\right)^2$$ (convolution theorem)



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10. Spectrum's Derivation

The derivative of the spectrum is the Fourier transform of the signal $-jtx(t)$.

$$tx(t) \rightarrow j\frac{dX(\omega)}{d\omega}$$

$$\frac{dX(\omega)}{d\omega} = \frac{d}{d\omega}\left(\int_{-\infty}^{\infty} x(t)\cdot e^{-j\omega t}dt\right) = \int_{-\infty}^{\infty} x(t)\cdot\frac{d}{d\omega}\left(e^{-j\omega t}\right)dt =$$

$$= \int_{-\infty}^{\infty} x(t)\cdot(-jt)\cdot e^{-j\omega t}dt$$

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11. CTFT of Real Signals. Properties.

(The Spectrum of the Even and Odd Parts of a Real Signal)

•The spectrum of a real and even signal is real and even.

•The spectrum of a real and odd signal is imaginary and odd.

$$x_e(t) \leftrightarrow \mathrm{Re}\{X\{\omega\}\} = X_E(\omega) \quad \text{and}$$
$$x_o(t) \leftrightarrow j\,\mathrm{Im}\{X(\omega)\} = jX_O(\omega)$$

Proof

- the real signal $x(t)$ with spectrum $X(\omega)$, complex :

$$X(\omega) = \underbrace{|X(\omega)| \cdot e^{j\Phi(\omega)}}_{\text{Polar form}} = \underbrace{\mathrm{Re}\{X(\omega)\} + j\,\mathrm{Im}\{X(\omega)\}}_{\text{Cartesian form}}$$

- Its complex conjugate real $x^*(t) \leftrightarrow X^*(-\omega)$

$$X^*(-\omega) = \underbrace{|X(-\omega)| \cdot e^{-j\Phi(-\omega)}}_{\text{Polar form}} = \underbrace{\mathrm{Re}\{X(-\omega)\} - j\,\mathrm{Im}\{X(-\omega)\}}_{\text{Cartesian form}}$$

•For real signals:

$$x^*(t) = x(t) \Rightarrow X^*(-\omega) = X(\omega)$$

•By identification:

$$|X(\omega)| = |X(-\omega)|; \quad \Phi(\omega) = -\Phi(-\omega);$$

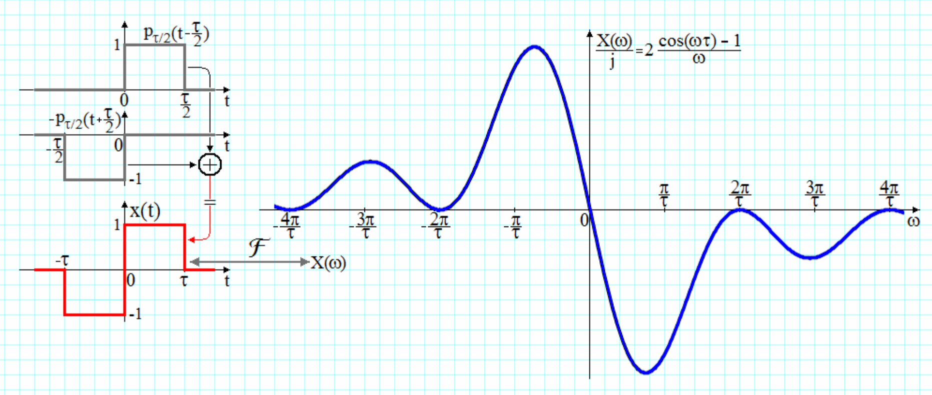
$$\mathrm{Re}\{X(\omega)\} = \mathrm{Re}\{X(-\omega)\}; \quad \mathrm{Im}\{X(\omega)\} = -\mathrm{Im}\{X(-\omega)\}.$$

Magnitude and real part of spectrum are **even functions**

Phase and imaginary part of spectrum are **odd functions** 43

Example - odd real signal

$$x(t) = p_{\frac{\tau}{2}}\left(t - \frac{\tau}{2}\right) - p_{\frac{\tau}{2}}\left(t + \frac{\tau}{2}\right) \leftrightarrow 2\frac{\sin\frac{\omega\tau}{2}}{\omega}\left(e^{-j\frac{\omega\tau}{2}} - e^{j\frac{\omega\tau}{2}}\right)$$



The spectrum of a real and odd signal is **imaginary and odd**

$$p_{\frac{\tau}{2}}(t) \leftrightarrow 2\frac{sin\frac{\omega\tau}{2}}{\omega}$$

time shifting

$$p_{\frac{\tau}{2}}\left(t-\frac{\tau}{2}\right) \leftrightarrow e^{-j\frac{\omega\tau}{2}}2\frac{sin\frac{\omega\tau}{2}}{\omega} \text{ and } p_{\frac{\tau}{2}}\left(t+\frac{\tau}{2}\right) \leftrightarrow e^{j\frac{\omega\tau}{2}}2\frac{sin\frac{\omega\tau}{2}}{\omega}$$

$$x(t) \leftrightarrow 2\frac{\sin\frac{\omega\tau}{2}}{\omega}\left(e^{-j\frac{\omega\tau}{2}} - e^{j\frac{\omega\tau}{2}}\right) = -2j\frac{1-\cos\omega\tau}{\omega}$$

Euler's relation
sin2(u) =1-cos (2u)

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12. A Parseval like theorem for signals from L^1

$$\int_{-\infty}^{\infty} \mathcal{F}^1\{x(\omega)\}(t)\,y(t)\,dt = \int_{-\infty}^{\infty} x(\omega)\mathcal{F}^1\{y(t)\}(\omega)\,d\omega$$

equivalent form:

$$\int_{-\infty}^{\infty} X(t)y(t)dt = \int_{-\infty}^{\infty} x(\omega)Y(\omega)d\omega$$

Fourier transform of the signal x(t) with the variable t

Signal x(t) with variable ω

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13. Relation Fourier Transform of a non-periodic signal & exponential Fourier series coefficients of the periodic signal

(see previous slides)

$$c_k = \frac{1}{T} X\left(k\omega_0\right), \quad \omega_0 = \frac{2\pi}{T}$$

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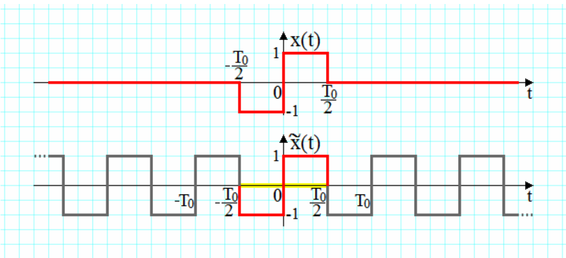
Example

$$x(t) = p_{\frac{T_0}{2}}\left(t - \frac{T_0}{2}\right) - p_{\frac{T_0}{2}}\left(t + \frac{T_0}{2}\right)$$



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the spectrum of the signal $x(t)$:

$$X(\omega) = -2j \frac{1 - cos\frac{\omega T_0}{2}}{\omega}$$

Applying the property 13:

$$c_k^x = -\frac{2j}{T_0} \cdot \frac{1 - \cos\frac{k\omega_0 T_0}{2}}{k\omega_0} = -2j \cdot \frac{1 - \cos k\pi}{k\pi}$$

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1) finite energy signals $x(t) \in L^1 \cap L^2$

The Fourier transform of a signal from $L^1 \cap L^2$ is from L^2

$$\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega = 2\pi \int_{-\infty}^{\infty} |x(t)|^2 dt$$

the **energy** of a signal in the frequency / time domain.
(**Parseval or Rayleigh** relation)

$|X(\omega)|^2$ = energy density.

using the L^2 norm:

$$\|X(\omega)\|_2^2 = 2\pi \|x(t)\|_2^2$$

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Proof

If $x(t) \in L^1 \cap L^2 \Rightarrow x^*(-t) \in L^1 \cap L^2$.

Their convolution belongs to L^1.

$$y(t) = x(t) * x^*(-t)$$

So, it has Fourier transform, $Y(\omega)$. from the convolution theorem :

$$Y(\omega) = X(\omega) \cdot X^*(-(-\omega)) = X(\omega) \cdot X^*(\omega) = |X(\omega)|^2$$

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We have:

$$y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} Y(\omega) \cdot e^{j\omega t} d\omega = \int_{-\infty}^{\infty} x(\tau) x^*(\tau - t) d\tau$$

for t=0:

$$y(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega = \int_{-\infty}^{\infty} x(\tau) x^*(\tau) d\tau$$

So:

$$\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega = 2\pi y(0) - \text{finite}$$

In consequence the function $X(\omega)$ belongs to L^2.

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2) finite energy signals $x(t) \in L^2 \setminus L^1$

•the Fourier transform of a finite energy signal:

Fourier transform in L2 space — convergence in mean square

$$\Im^2\{x(t)\}(\omega) = \underset{\tau\to\infty}{\text{l.i.m.}} \int_{-\tau}^{\tau} x(t)\cdot e^{-j\omega t}\,dt$$

•The L^2 norm of the Fourier transform :

$$\left\|\Im^2\{x(t)\}(\omega)\right\|_2 = \lim_{\tau\to\infty}\left\|\int_{-\tau}^{\tau} x(t)\cdot e^{-j\omega t}\,dt\right\|_2$$

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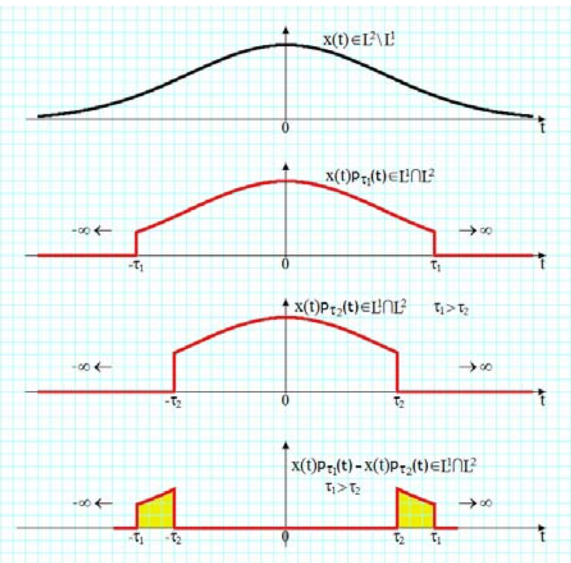

•Truncation x(t) by multiplication with $p_\tau(t)$

⇒approximation of $x(t) \in L^1 \cap L^2$

Two approximations.

•The "better" one - longer support.

•The other -an approximation of the first.

•The error tends to zero if the two durations tend to infinity.

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the approximation error $y(t)=x(t)\left(p_{\tau_1}(t)-p_{\tau_2}(t)\right)\in L^1\cap L^2$

$$\left\|\Im^1\{x(t)p_{\tau_1}(t)\}(\omega)-\Im^1\{x(t)p_{\tau_2}(t)\}(\omega)\right\|_2^2=$$

$$=2\pi\left\|x(t)p_{\tau_1}(t)-x(t)p_{\tau_2}(t)\right\|_2^2=2\pi\int_{-\tau_1}^{-\tau_2}|x(t)|^2dt+2\pi\int_{\tau_2}^{\tau_1}|x(t)|^2dt$$ Parseval

limit for $\tau_{1,2}\to\infty,\ \tau_1>\tau_2$

$$\lim_{\substack{\tau_{1,2}\to\infty\\ \tau_1>\tau_2}}\left\|\int_{-\tau_1}^{\tau_1}x(t)\cdot e^{-j\omega t}dt-\int_{-\tau_2}^{\tau_2}x(t)\cdot e^{-j\omega t}dt\right\|_2^2=0$$

$$\int_{-\tau}^{\tau}x(t)\cdot e^{-j\omega t}dt\to\Im^1\{x(t)p_\tau(t)\}$$ in **mean square**.

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Plâncherel's Theorem

The Fourier transform definition of a finite energy signal already given can be found under the name of Plâncherel's Theorem:

If $x(t)\in L^2$ then:

i) it exists $X(\omega)=\underset{\tau\to\infty}{l.i.m}\int_{-\tau}^{\tau}x(t)e^{-j\omega t}dt\ ,\ \forall\omega\in\mathbf{R},$

ii) for $\forall t\in R$ the following equality holds:

$$x(t)=\underset{R\to\infty}{l.i.m}\frac{1}{2\pi}\int_{-R}^{R}X(\omega)e^{j\omega t}d\omega$$

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Plâncherel's Theorem

i) Plâncherel's theorem shows that the Fourier transform of any finite energy signal belongs to L^2.

ii) The Fourier transform on L^2 is a particular case of the Fourier transform on L^1. All the properties of the Fourier transform on L^1 are verified by the Fourier transform on L^2.

The Parseval's relation - proved for signals in $L^1 \cap L^2$.

It is not verified by signals in $L^2 - L^1$

iii) The Parseval's relation can be generalized on L^2, in the form:

$$\langle x(t), y(t) \rangle = \frac{1}{2\pi} \langle X(\omega), Y(\omega) \rangle$$

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the definition of the scalar product on L^2:

$$\int_{-\infty}^{\infty} x(t) y^*(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) Y^*(\omega) d\omega$$

If the two signals are equal :

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

Parseval's relation.

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14. Spectrum's Convolution

- The convolution of the Fourier transforms $X(\omega)$ and $Y(\omega)$ gives the Fourier transform of the product $x(t)\,y(t)$ multiplied by 2π.

$$X(\omega)*Y(\omega)=2\pi\mathcal{F}\{x(t)\,y(t)\}(\omega)$$

- The convolution of two finite energy signals is of finite energy. Convolving two finite energy spectra $X(\omega)$ and $Y(\omega)$ $\Rightarrow$ a finite energy spectrum $Z(\omega)$

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$$Z(\omega)=X(\omega)*Y(\omega)=\int_{-\infty}^{\infty}X(u)\,Y(\omega-u)\,du$$

$$=2\pi\int_{-\infty}^{\infty}X(u)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}y(t)\cdot e^{-j(\omega-u)t}dt\right]du$$

$$Z(\omega)=2\pi\int_{-\infty}^{\infty}y(t)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}X(u)\cdot e^{j\omega u}du\right]\cdot e^{-j\omega t}dt$$

Inverse Fourier transform of X = x(t)

$$Z(\omega)=2\pi\int_{-\infty}^{\infty}x(t)y(t)\cdot e^{-j\omega t}dt$$

the Fourier transform of the product $x(t)\cdot y(t)$.

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15. Duality

The inverse Fourier transform : $2\pi x(t) = \int_{-\infty}^{\infty} X(\omega) \cdot e^{j\omega t} d\omega$

For $t = -t$:

$$2\pi x(-t) = \underbrace{\int_{-\infty}^{\infty} X(\omega) \cdot e^{-j\omega t} d\omega}_{\text{Fourier transform of } X(\omega).} \text{, duality}$$

Applying two times the Fourier transform ⇒a reversed variant of the original signal weighted by 2π.

$$2\pi x(-t) = \Im^2 \{\Im^2 \{x(t)\}(\omega)\}(t)$$

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15. Duality

$$2\pi x(-t) = \int_{-\infty}^{\infty} X(\omega) \cdot e^{-j\omega t} d\omega$$

double change of variables: $t \to \omega$ and $\omega \to t$

$$2\pi x(-\omega) = \int_{-\infty}^{\infty} X(t) \cdot e^{-j\omega t} dt = \Im\{X(t)\}(\omega)$$, another form of duality.

Using the two forms of duality we can compute the spectrum of a signal.

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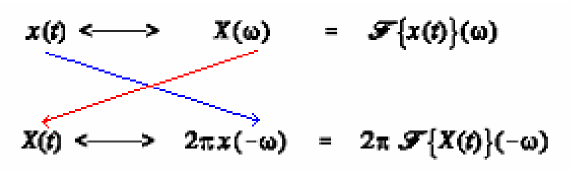

- Start from a known pair $(x(t), X(\omega))$
- What is the spectrum of the signal $X(t)$?
- Change the variable and constants of time with variable and constants of frequency and vice versa
- obtain the corresponding pair $(X(t), 2\pi x(-\omega))$.

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The Fourier Transform of signals Temporal window

the spectrum

$$p_\tau(t) = \sigma(t+\tau) - \sigma(t-\tau) \leftrightarrow 2\frac{\sin \omega\tau}{\omega} \ .$$

In this case, $x(t) = p_\tau(t)$ and $X(\omega) = 2\frac{\sin \omega\tau}{\omega}$.

Changing the variables $t \leftrightarrow \omega$ and constants $\tau \leftrightarrow \omega_0$

$$X(t) = 2\frac{\sin \omega_0 t}{t} \longrightarrow 2\pi x(-\omega) = 2\pi p_{\omega_0}(\omega) \ .$$

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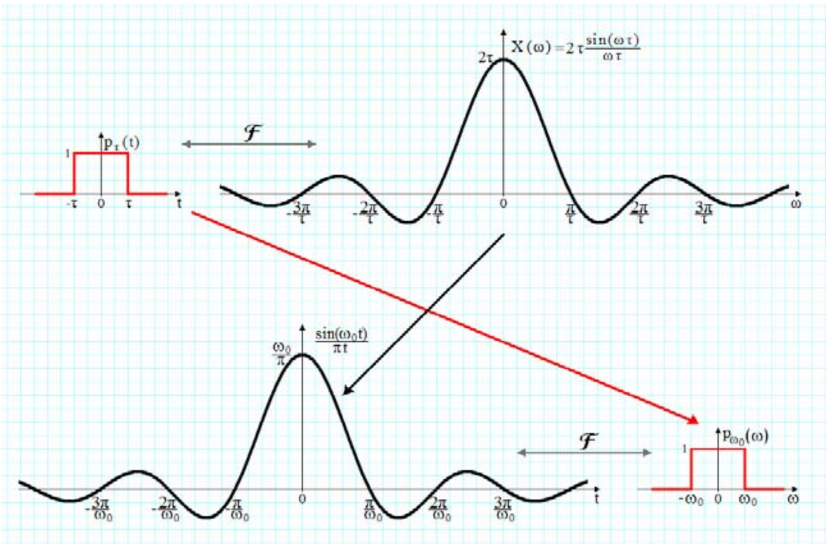

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Symmetric triangular signal

the spectrum

$$tri_T(t) = T\left(1 - \frac{|t|}{T}\right) p_T(t) \leftrightarrow \left(\frac{\sin\frac{\omega T}{2}}{\frac{\omega}{2}}\right)^2 .$$

$$x(t) = tri_T(t) \text{ and } X(\omega) = \left(\frac{\sin\frac{\omega T}{2}}{\frac{\omega}{2}}\right)^2 .$$

$$X(t) = \left(\frac{\sin\frac{\omega_0 t}{2}}{\frac{t}{2}}\right)^2 \longrightarrow 2\pi x(-\omega) = 2\pi \cdot tri_{\omega_0}(\omega).$$

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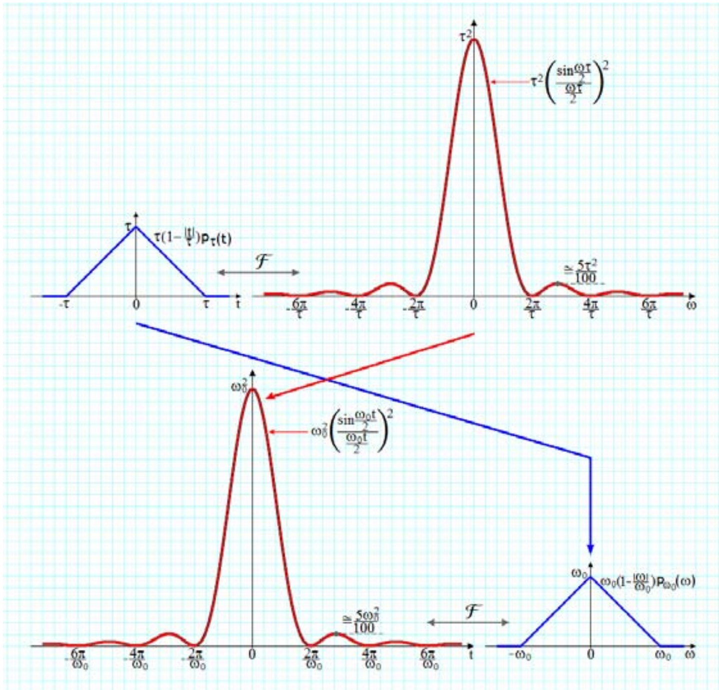

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Decreasing causal exponential

$x(t) = e^{-\omega_0 t}\sigma(t)$ with $\omega_0 > 0$.

$$X(\omega) = \int_0^{\infty} e^{-\omega_0 t} \cdot e^{-j\omega t} dt = \int_0^{\infty} e^{-(\omega_0 + j\omega)t} dt = -\frac{1}{\omega_0 + j\omega} e^{-(\omega_0 + j\omega)t} \Big|_0^{\infty} = \frac{1}{\omega_0 + j\omega}$$

$$|X(\omega)| = \left| \frac{1}{\omega_0 + j\omega} \right| = \frac{1}{|\omega_0 + j\omega|} = \frac{1}{\sqrt{\omega_0^2 + \omega^2}}$$

$$\Phi(\omega) = arg\{X(\omega)\} = arg\left\{ \frac{1}{\omega_0 + j\omega} \right\} = arg\{1\} - arg\{\omega_0 + j\omega\} = -arctg\left(\frac{\omega}{\omega_0} \right)$$

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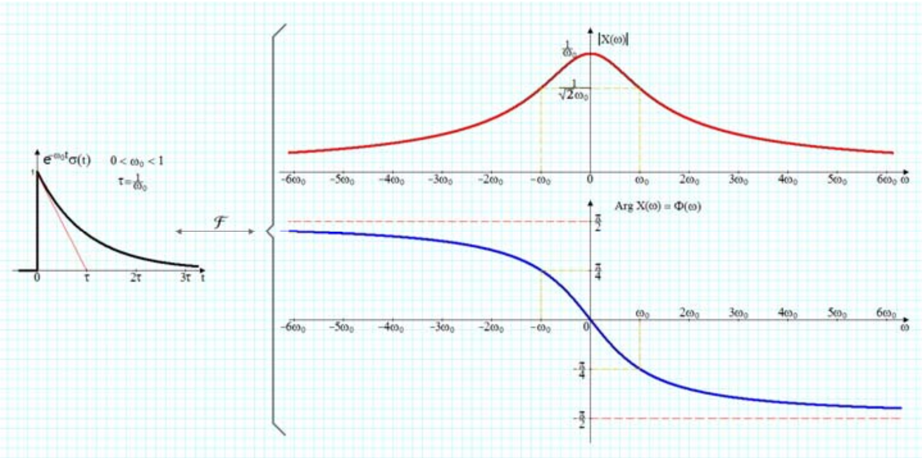
$$|X(\omega)| = \frac{1}{\sqrt{\omega_0^2 + \omega^2}}$$



$$\Phi(\omega) = -arctg\left(\frac{\omega}{\omega_0}\right)$$

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Decreasing non-causal exponential

$$x(t) = e^{-\omega_0 t}\sigma(t) \text{ with } \omega_0 > 0.$$

$$\breve{x}(t) = e^{\omega_0 t}\sigma(-t) \text{ with } \omega_0 > 0.$$

$$\breve{X}(\omega) = X(-\omega) = \frac{1}{\omega_0 - j\omega}.$$

$$|\breve{X}(\omega)| = |X(\omega)| = \frac{1}{\sqrt{\omega_0^2 + \omega^2}} \quad arg\{\breve{X}(\omega)\} = arg\left\{\frac{1}{\omega_0 - j\omega}\right\} = arctg\left(\frac{\omega}{\omega_0}\right).$$

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Symmetric decreasing exponential

$$x_s(t) = e^{-\omega_0|t|} = \begin{cases} e^{-\omega_0 t}, & t \geq 0 \\ e^{\omega_0 t}, & t \leq 0 \end{cases}; \ 0 < \omega_0 < 1$$

$$x_s(t) = x(t) + \breve{x}(t).$$

$$X_s(\omega) = X(\omega) + \breve{X}(\omega) = \frac{1}{\omega_0 + j\omega} + \frac{1}{\omega_0 - j\omega} = \frac{2\omega_0}{\omega_0^2 + \omega^2}$$

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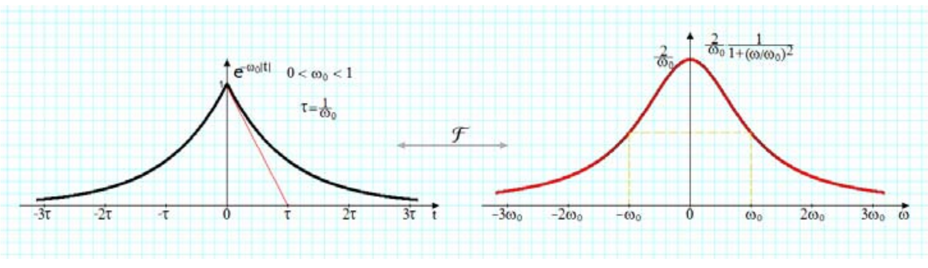

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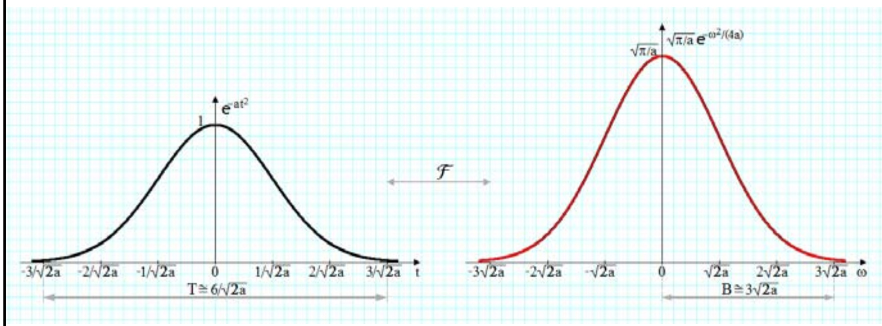
Gaussian signal

$$e^{-at^2} \leftrightarrow \sqrt{\frac{\pi}{a}} e^{-\frac{1}{4a}\omega^2} \quad , \quad a > 0.$$

The spectrum of a Gaussian signal is Gaussian



The Fourier Transform of Distributions

1) The spectrum of the Dirac's distribution

for any test function $\varphi(t)$:

$$\int_{-\infty}^{\infty} \varphi(t)\delta(t)dt = \varphi(0) \text{ or } \int_{-\infty}^{\infty} \varphi(t)\delta(-t)dt = \varphi(0)$$

the Dirac's distribution is even.

$$\varphi(t) = e^{-j\omega t} \Rightarrow \int_{-\infty}^{\infty} \delta(t) \cdot e^{-j\omega t} dt = \Im\{\delta(t)\}(\omega) = 1$$

Hence, we have obtained:

$$\delta(t) \leftrightarrow 1(\omega)$$

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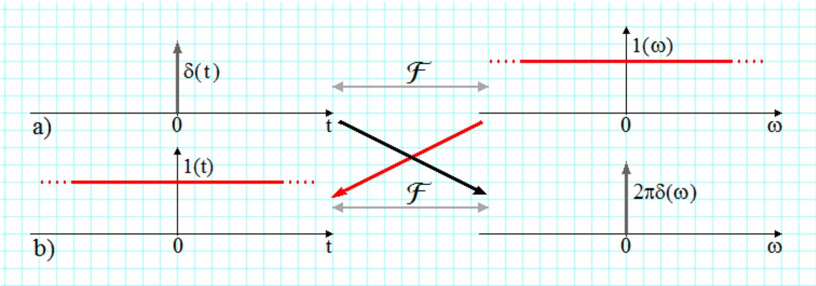
2) The spectrum of the constant $1(t)$

duality$\Rightarrow$ $1(t) \leftrightarrow 2\pi\delta(-\omega) = 2\pi\delta(\omega)$

$$c \leftrightarrow 2\pi c\delta(\omega)$$



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3) The spectrum of the unit step σ(t)

$$u(t) = \frac{1}{2}\operatorname{sgn} t \;\text{ and }\; v(t) = \frac{1}{2}.$$

$$\Im\{u'(t)\}(\omega) = \Im\{\delta(t)\}(\omega) = 1.$$

$$\Im\{u(t)\}(\omega) = \frac{\Im\{u'(t)\}(\omega)}{j\omega} = \frac{1}{j\omega} \Rightarrow \sigma(t) = u(t) + v(t)$$

$$\Im\{\sigma(t)\}(\omega) = \frac{1}{j\omega} + \pi\delta(\omega)$$

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4) The spectrum of *sgn*(*t*)

$$\Im\{sgn\,t\} = 2\Im\{u(t)\} = \frac{2}{j\omega}$$

$$\operatorname{sgn} t = \begin{cases} 1, t > 0 \\ 0, t = 0 \\ -1, t < 0 \end{cases}$$

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5) The spectrum of the signal 1/(π*t*)

$$sgn\,t \leftrightarrow \frac{2}{j\omega}$$

$$\frac{2}{jt} \leftrightarrow 2\pi\, sgn(-\omega) = -2\pi\, sgn\,\omega \qquad \text{(duality)}$$

$$\frac{1}{\pi t} \leftrightarrow -j \operatorname{sgn}\omega = \begin{cases} -j, & \omega > 0 \\ 0, & \omega = 0 \\ j, & \omega < 0 \end{cases}$$

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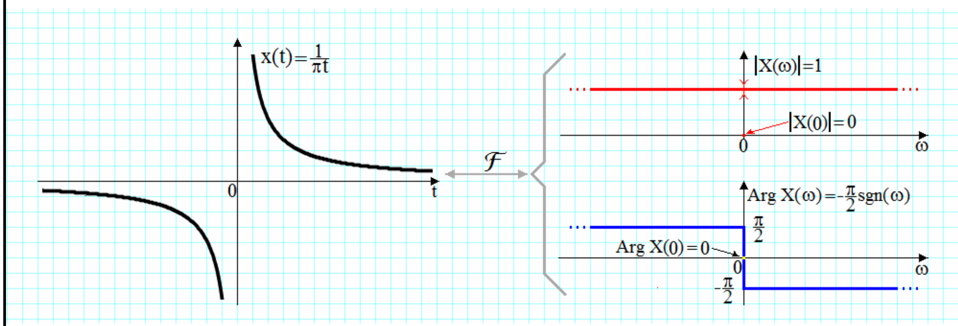


6) Fourier Transform of the integral of a signal having DC component, $X(0)\neq 0$

$$\int_{-\infty}^{t} x(\tau)d\tau \leftrightarrow \frac{X(\omega)}{j\omega} + \pi X(0)\delta(\omega)$$

- Proof:

$$y(t) = \int_{-\infty}^{t} x(\tau)d\tau = \int_{-\infty}^{\infty} x(\tau)\sigma(t\text{-}\tau)d\tau$$

$$Y(\omega) = X(\omega)\cdot\Im\{\sigma(t)\}(\omega) = X(\omega)\left[\frac{1}{j\omega} + \pi\delta(\omega)\right] = \frac{X(\omega)}{j\omega} + \pi X(\omega)\delta(\omega)$$

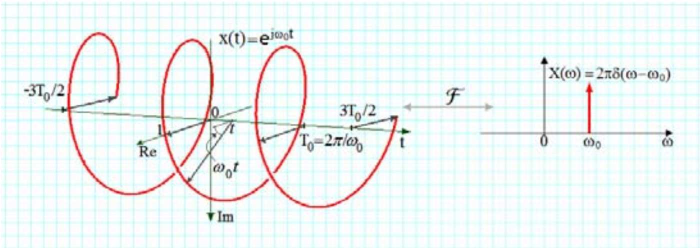
$$X(\omega)\cdot\delta(\omega) = X(0)\cdot\delta(\omega)$$

7) The spectrum of the complex exponential

$1(t) \leftrightarrow 2\pi\delta(\omega)$

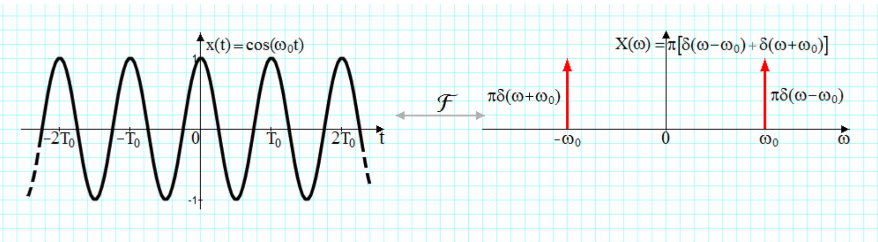
Modulation:

$$e^{j\omega_0 t} \leftrightarrow 2\pi\delta(\omega - \omega_0)$$



8) The spectrum of $\cos\omega_0 t$

$$\cos\omega_0 t = \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2} \leftrightarrow \pi\left[\delta(\omega\text{-}\omega_0) + \delta(\omega + \omega_0)\right]$$



Spectrum for a limited cosine of duration 2 τ

$$x(t) = \cos\omega_0 t \cdot p_\tau(t)$$

$$X(\omega) = \frac{1}{2\pi}\Im\{\cos\omega_0 t\} * \Im\{p_\tau(t)\}$$

$$= \frac{1}{2\pi}\left[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right] * 2\frac{\sin\omega\tau}{\omega}$$

$$X(\omega) = \tau\left[\frac{\sin(\omega-\omega_0)\tau}{(\omega-\omega_0)\tau} + \frac{\sin(\omega+\omega_0)\tau}{(\omega+\omega_0)\tau}\right]$$

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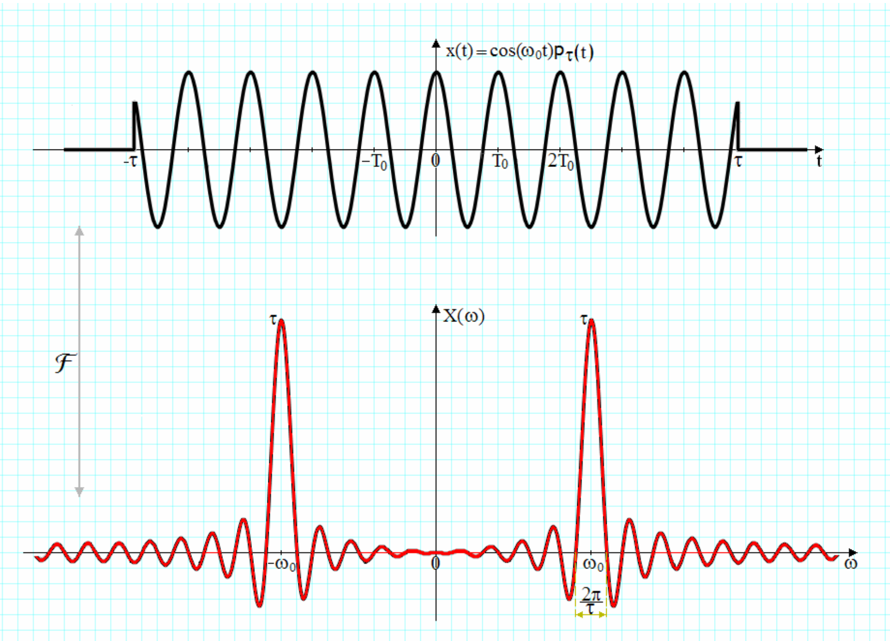
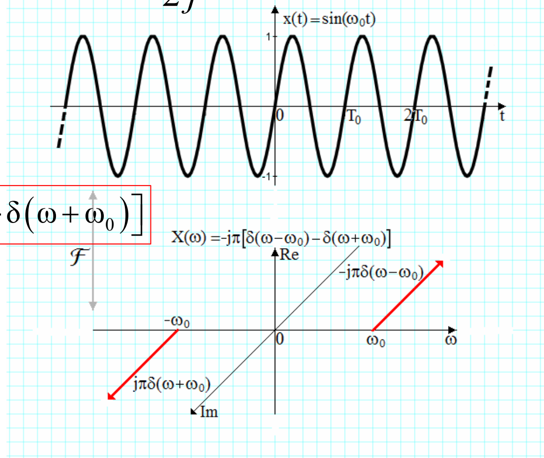


9) The spectrum of $\sin\omega_0 t$

$$\sin\omega_0 t = \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j} \leftrightarrow \frac{2\pi\delta(\omega\text{-}\omega_0) - 2\pi\delta(\omega+\omega_0)}{2j}$$

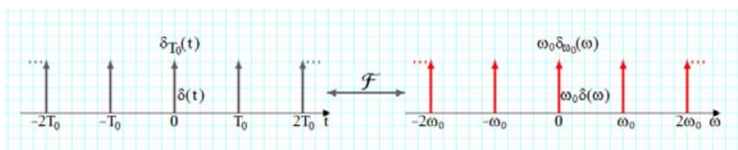
$$\sin\omega_0 t \leftrightarrow -j\pi\left[\delta(\omega\text{-}\omega_0) - \delta(\omega+\omega_0)\right]$$



Fourier transform of periodic signals

- The periodic signal $y(t)$ =convolution of its restriction at one period, $x(t)$ and the periodic Dirac's distribution

$$y(t) = x(t) * \delta_{T_0}(t) \qquad \delta_{T0}(t) = \frac{1}{T_0}\sum_{k=-\infty}^{\infty} e^{-jk\omega_0 t} \leftrightarrow \omega_0\delta_{\omega 0}(\omega)$$



- the Fourier transform of periodic $\delta_{T0}(t)$ with period T_0

  -proportional with the periodic $\delta_{\omega 0}(t)$ with period ω_0.

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$$\delta_{T0}(t) = \frac{1}{T_0}\sum_{k=-\infty}^{\infty} e^{-jk\omega_0 t} = \sum_{k=-\infty}^{\infty} \delta(t - kT_0)$$

$$\Im\{\delta_{T_0}(t)\}(\omega) = \sum_{k=-\infty}^{\infty} \Im\{\delta(t\text{-}kT_0)\}(\omega)$$

But: $\delta(t) \leftrightarrow 1$ and $\delta(t\text{-}t_0) \leftrightarrow e^{-j\omega t_0}$

So: $$\Im\{\delta_{T_0}(t)\}(\omega) = \sum_{k=-\infty}^{\infty} e^{-jk\omega T_0}$$

$$\delta_{T0}(t) = \frac{1}{T_0}\sum_{k=-\infty}^{\infty} e^{-jk\omega_0 t} \xrightarrow{\text{Variable and constant changes}} \delta_{\omega 0}(\omega) = \frac{1}{\omega_0}\underbrace{\sum_{k=-\infty}^{\infty} e^{-jk\omega T_0}}_{\Im\{\delta_{T_0}(t)\}(\omega)}$$

$t \leftrightarrow \omega$ and $T_0 \leftrightarrow \omega_0$

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$$Y(\omega) = X(\omega)\cdot\Im\{\delta_{T_0}(t)\}(\omega) = X(\omega)\cdot\omega_0\cdot\delta_{\omega 0}(\omega)$$

$$\boxed{Y(\omega) = \frac{2\pi}{T_0}\sum_{k=-\infty}^{\infty} X(k\omega_0)\cdot\delta(\omega - k\omega_0)}$$

We have the relation Fourier coefficients of the periodic signal $y(t)$ with the Fourier transform of the non-periodic signal $x(t)$:

$$c_k^y = \frac{X(k\omega_0)}{T_0}$$

The Fourier transform of the periodic signal is:

$$\boxed{Y(\omega) = \sum_{k=-\infty}^{\infty} 2\pi\cdot c_k^y\cdot\delta(\omega - k\omega_0)}$$

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The effect of signal’s truncation

$$x(t)=\frac{\sin\omega_0 t}{\pi t}\leftrightarrow p_{\omega_0}(\omega)$$

$$\frac{\sin\omega_0 t}{\pi t}p_\tau(t)\leftrightarrow\frac{1}{2\pi}p_{\omega_0}(\omega)*2\frac{\sin\omega\tau}{\omega}=\hat{X}(\omega)$$

$$\hat{X}(\omega)=\frac{1}{\pi}\int_{-\infty}^{\infty}p_{\omega_0}(\omega-u)\frac{\sin u\tau}{u}du=\frac{1}{\pi}\int_{\omega-\omega_0}^{\omega+\omega_0}\frac{\sin u\tau}{u}du=\frac{1}{\pi}\mathrm{Si}(y)\Big|_{\tau(\omega-\omega_0)}^{\tau(\omega+\omega_0)}$$

$$=\frac{1}{\pi}\mathrm{Si}\left[\tau(\omega+\omega_0)\right]-\frac{1}{\pi}\mathrm{Si}\left[\tau(\omega-\omega_0)\right]$$

$\hat{X}(\omega)$ converges in mean square to $X(\omega)=p_{\omega_0}(\omega)$:

$$\underset{\tau\to\infty}{l.i.m.}\,\mathcal{F}\left\{x(t)\,p_\tau(t)\right\}=\mathcal{F}\left\{x(t)\right\}$$

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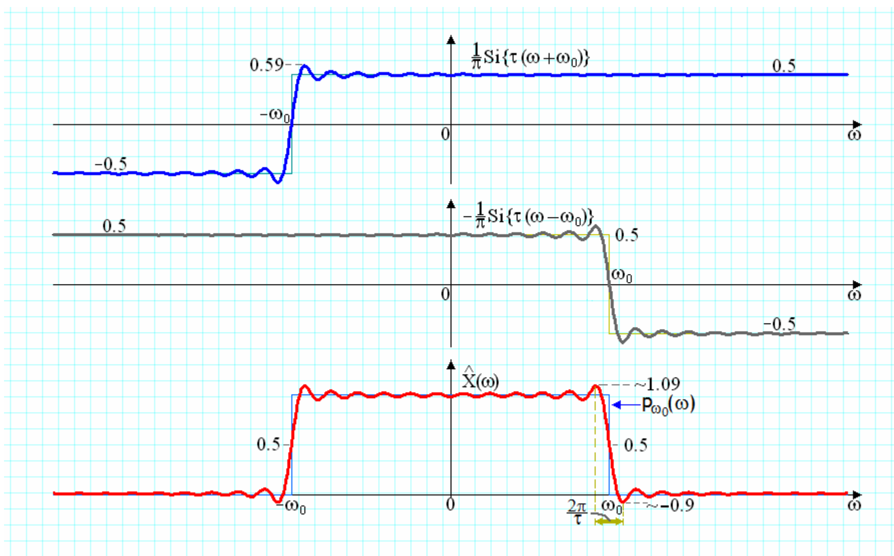

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The effect of the spectrum's truncation on the reconstructed signal

Rectangular pulse: $x(t) = p_\tau(t) \leftrightarrow 2\dfrac{\sin\omega\tau}{\omega} = X(\omega);$

Truncated spectrum from $-\omega_0$ to ω_0: $\hat{x}(t) = ? \leftrightarrow \hat{X}(\omega) = 2\dfrac{\sin\omega\tau}{\omega}p_{\omega_0}(\omega)$

$\dfrac{\sin\omega_0 t}{\pi t} \leftrightarrow p_{\omega_0}(\omega)$ and $p_\tau(t) \leftrightarrow 2\dfrac{\sin\omega\tau}{\omega}$

$\Rightarrow \hat{x}(t) = \dfrac{\sin\omega_0 t}{\pi t} * p_\tau(t) \leftrightarrow 2\dfrac{\sin\omega\tau}{\omega}p_{\omega_0}(\omega)$

$\dfrac{\sin\omega_0 t}{\pi t} p_\tau(t) \leftrightarrow \dfrac{1}{\pi}\mathrm{Si}\left[\tau(\omega+\omega_0)\right] - \dfrac{1}{\pi}\mathrm{Si}\left[\tau(\omega-\omega_0)\right]$

Duality:

$\dfrac{1}{\pi}\mathrm{Si}\left[\omega_0(t+\tau)\right] - \dfrac{1}{\pi}\mathrm{Si}\left[\omega_0(t-\tau)\right] \leftrightarrow 2\pi\dfrac{\sin(-\omega\tau)}{-\pi\omega}p_{\omega_0}(-\omega) = 2\dfrac{\sin(\omega\tau)}{\omega}p_{\omega_0}(\omega)$

So, $\hat{x}(t) = \dfrac{1}{\pi}\mathrm{Si}\left[\omega_0(t+\tau)\right] - \dfrac{1}{\pi}\mathrm{Si}\left[\omega_0(t-\tau)\right]$

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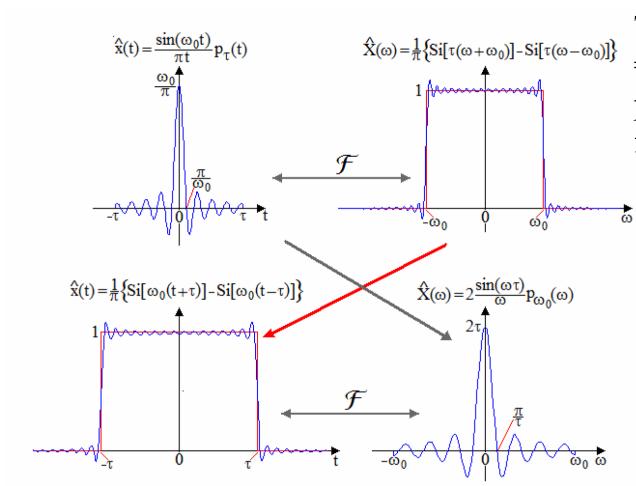

Truncation in time => Gibbs phenomenon in frequency

Truncation in frequency => Gibbs phenomenon in time

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Repartition

Different energy concentration measures. The repartition of a random variable X is described by its probability density function $f_X(x)$: $f_X(x) \geq 0$ and $\int_{-\infty}^{\infty} f_X(x)dx = 1$

i) Mean $\mu_X = E\{X\} = \int_{-\infty}^{\infty} x f_X(x)dx;$

ii) Power $E\{X^2\} = \int_{-\infty}^{\infty} x^2 f_X(x)dx;$

iii) Variance $Var\{X\} = E\left\{(X-\mu_X)^2\right\} = \int_{-\infty}^{\infty}(x-\mu_X)^2 f_X(x)dx$

iv) Standard deviation $\sigma_X = \sqrt{Var(X)}.$

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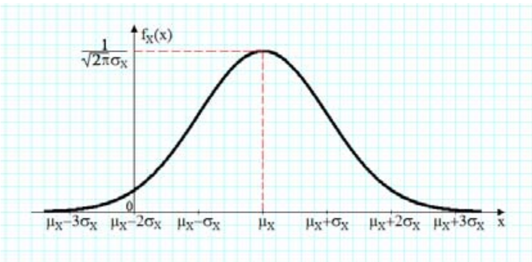
Example: Gaussian (normal) repartition

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma_X} e^{-\frac{(x-\mu_X)^2}{2\sigma_X^2}}$$

μ_X -mean

σ_X -standard deviation



$$\frac{1}{\sqrt{2\pi}\sigma_X}\int_{-\infty}^{\infty} e^{-\frac{(x-\mu_X)^2}{2\sigma_X^2}} dx = 1$$

$\mu_X = 0, \sigma_X = 1$

$$\Rightarrow \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx = 1$$

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Signal energy's distribution in time

The energy of a signal $x(t)$: $W = \int_{-\infty}^{\infty} |x(t)|^2 dt$

$\frac{|x(t)|^2}{W}$ energy distribution function, in time.

- Average time t_c - the energy of the signal is concentrated with the dispersion of σ_t^2 =**time spreading**

$$t_c = \frac{\int_{-\infty}^{\infty} t|x(t)|^2 dt}{\int_{-\infty}^{\infty} |x(t)|^2 dt} \qquad \sigma_t^2 = \frac{\int_{-\infty}^{\infty} (t - t_c)^2 |x(t)|^2 dt}{\int_{-\infty}^{\infty} |x(t)|^2 dt}$$

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Signal energy's distribution in frequency

The energy of signal $x(t)$, spectrum $X(\omega)$: $W = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$

$\frac{|X(\omega)|^2}{W}$ energy distribution function, in frequency.

Average frequency ω_c -the energy of the signal is concentrated with dispersion of σ_ω^2 =**frequency spreading**,

$$\omega_c = \frac{\int_{-\infty}^{\infty} \omega|X(\omega)|^2 d\omega}{\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega} \qquad \sigma_\omega^2 = \frac{\int_{-\infty}^{\infty} (\omega - \omega_c)^2 |X(\omega)|^2 d\omega}{\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega}$$

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The values of standard deviations σ_t and σ_ω give us information about the effective duration and the effective bandwidth of the signal $x(t)$.

The Heisenberg-Gabor uncertainty principle

If σ_t and σ_ω can be defined, then for any signal we have:

$$\sigma_t \sigma_\omega \geq \frac{1}{2}$$

The sign equal appears if and only if $x(t)$ is a Gaussian signal.

-there are not signals with perfect concentration of energy in the time-frequency plane

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Example: Gaussian signal

$$x(t) = e^{-at^2} \leftrightarrow X(\omega) = \sqrt{\frac{\pi}{a}} e^{-\frac{1}{4a}\omega^2}$$

$$t_c = 0; \qquad \sigma_t^2 = \frac{1}{4a}; \quad \omega_c = 0 \qquad \sigma_\omega^2 = a$$

The product $\sigma_t \sigma_\omega = \frac{1}{2}$. The Heisenberg-Gabor inequality is satisfied with the equal sign.

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The energy in the (time) interval $\left[-3\sigma_t, 3\sigma_t\right] = \left[-\frac{3}{2\sqrt{a}}, \frac{3}{2\sqrt{a}}\right]$

$$W_{6\sigma} = \int_{-\frac{3}{2\sqrt{a}}}^{\frac{3}{2\sqrt{a}}} e^{-2at^2} dt = 0.9974 \frac{\sqrt{2\pi}}{2\sqrt{a}} \qquad \frac{W_{6\sigma}}{W} = 99.74\%$$

The energy in the bandwidth $\left[0, 3\sigma_\omega\right]$

$$W'_{6\sigma} = \frac{1}{2\pi} \int_{-3\sqrt{a}}^{3\sqrt{a}} \frac{\pi}{a} e^{-\frac{\omega^2}{2a}} d\omega = 0.9974 \frac{\sqrt{2\pi}}{2\sqrt{a}}; \qquad \frac{W'_{6\sigma}}{W} = 99.74\%$$

Signal duration $T = \frac{3}{\sqrt{a}}$; its bandwidth $B_\omega = 3\sqrt{a}$

$\Rightarrow$ product duration-bandwidth $TB_\omega = 9$ for 99.74%energy

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Remarks: i)Interpretation of Heisenberg-Gabor inequality $\sigma_t \sigma_\omega = \frac{1}{2}$

If the signal duration σ_t increases $\Rightarrow$ bandwidth σ_ω decreases. Example: the time-scaling property. For a fixed duration, the spectral standard deviation is

$$\sigma_\omega = \frac{C}{\sigma_t} \geq \frac{1}{2\sigma_t}$$

Between all the signals with the same duration, the Gaussian one has minimum bandwidth. Reciprocically, between all the signals with the same bandwidth , the Gaussian one has minimum duration. The Gaussian signal is ideal for telecommunications transmission: at an imposed bandwidth it offers the highest transmission speed. Sometimes, t**he values σ_t and σ_ω can't be computed.**

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ii) The signal $x(t)=e^{-\omega_0 t}\sigma(t) \leftrightarrow X(\omega)=\frac{1}{\omega_0+j\omega}$

$$\int_{-\infty}^{\infty} t\left|x(t)\right|^2 dt=\frac{1}{(2\omega_0)^2};\ W=\int_{-\infty}^{\infty}\left|x(t)\right|^2 dt=\frac{1}{2\omega_0}\Rightarrow\ t_C=\frac{1}{(2\omega_0)^2}2\omega_0=\frac{1}{2\omega_0}$$

$$\int_{-\infty}^{\infty}\left(t-\frac{1}{2\omega_0}\right)^2 e^{-2\omega_0 t}\sigma(t)\,dt=\frac{1}{8\omega_0^3};\ \sigma_t=\frac{1}{2\omega_0}$$

$$\left|X(\omega)\right|^2=\frac{1}{\omega_0^2+\omega^2}\ \text{(even function)}\Rightarrow\omega_C=0\Rightarrow\ \sigma_\omega^2=\frac{\int_{-\infty}^{\infty}\omega^2\left|X(\omega)\right|^2 d\omega}{\int_{-\infty}^{\infty}\left|X(\omega)\right|^2 d\omega};$$

$$\int_{-\infty}^{\infty}\omega^2\left|X(\omega)\right|^2 d\omega=\int_{-\infty}^{\infty}\frac{\omega^2}{\omega_0^2+\omega^2}d\omega=\left(\omega-\omega_0 arctg\left(\frac{\omega}{\omega_0}\right)\right)\Bigg|_{-\infty}^{\infty}=\infty$$

$\Rightarrow \sigma_\omega$ can't be defined

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In the interval $[0,T]$ the energy of the signal is determined with:

$$W_T=\int_0^T\left|x(t)\right|^2 dt=\int_0^T e^{-2\omega_0 t}dt=\left(1-e^{-2\omega_0 t}\right)W$$

If we impose $W_T=0{,}995W$ then it result $T=\frac{2{,}65}{\omega_0}$.

The energy in the frequency band $[0,\ B_\omega]$ is computed with:

$$W_{B_\omega}=\frac{1}{2\pi}\int_{-B_\omega}^{B_\omega}\left|X(\omega)\right|^2 d\omega=\frac{2}{\pi}arctg\frac{B_\omega}{\omega_0}W.$$

If we impose $W_{B_\omega}=0{,}995W$ then it result $B_\omega\cong 127{,}3\omega_0$.

So, for this signal the product TB_ω is of 337,3.

Such a signal, at a duration T imposed has a very large bandwidth.

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iii) The signal temporal window, $p_\tau(t)$, with $T = 2\tau$. All the energy of the signal is contained in this duration.

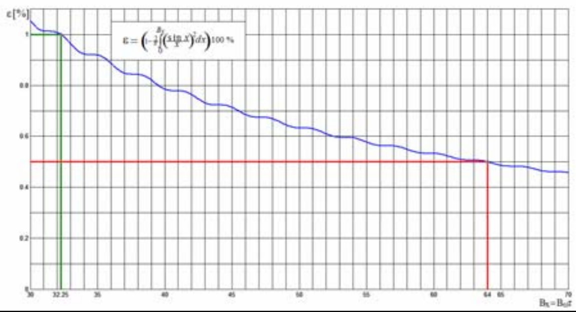
$$W_{B_\omega} = \frac{1}{2\pi}\int_{-B_\omega}^{B_\omega} |X(\omega)|^2 d\omega \overset{\omega\tau = x}{=} \frac{2\tau}{\pi}\int_{-B_x}^{B_x}\left(\frac{\sin x}{x}\right)^2 dx.$$

The energy not contained in the frequency interval $[0, B_\omega]$ is:

$$\varepsilon = \frac{W - W_{B_\omega}}{W} = \frac{\pi - \int_{-B_x}^{B_x}\left(\frac{\sin x}{x}\right)^2 dx}{\pi}$$

For :

$$W_{B_\omega} = 0{,}995W$$



the duration-bandwidth product is 130. At the same duration the rectangular pulse has a smaller bandwidth than the exponential.

Special problems regarding signals
i) Band-limited signals

The band-limited signals have infinite duration.

They respect the Bernstein's theorem.

A band-limited signal bounded by M has all the derivatives bounded :

$$\left|x^{(k)}(t)\right| \le \omega_M^k M$$

- signal with slow variation.

They have compact spectrum's support, $[-\omega_M, \omega_M]$.

$x(t) \leftrightarrow X(\omega) = X(\omega)\, p_{\omega_M}(\omega)$ With the notation $z = t + ju$, we have:

$$x(t) = \frac{1}{2\pi} \int_{-\omega_M}^{\omega_M} X(\omega) e^{j\omega t} d\omega.$$

$$x(z) = \frac{1}{2\pi} \int_{-\omega_M}^{\omega_M} X(\omega) e^{j\omega z} d\omega,$$

is an integer function, derivable in all the complex plane and hence also on the real axis, t,

$$x^{(k)}(t) = \frac{1}{2\pi} \int_{-\omega_M}^{\omega_M} X(\omega)(j\omega)^k e^{j\omega t} d\omega \;\forall k \in N$$

It follows that $x(t)$ can't vanish on any compact and hence its support has infinte length.

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ii) Causal Signals and the Paley-Wiener Theorem

The signal x(t) is causal **if and only if** the integral:

$$I = \int_{-\infty}^{\infty} \frac{\left|\log\left|X(\omega)\right|\right|}{1+\omega^2} d\omega$$

is convergent.

The spectrum can be zero, in a countable set of points, having a null Lebesque measure.

The causal signals are non band-limited.

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