Fourier Series. Spectral Analysis of Periodic Signals

The response of continuous-time linear invariant systems to the complex exponential with unitary magnitude

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- response of a continuous-time LTI system at a certain signal:
  - differential equation
  - convolution product of the input signal with impulse response.
- input signal **periodic:** decompose into a series of simpler components,
  - response of the system at each component
  - synthesis of partial responses.
- frequency domain: Fourier series

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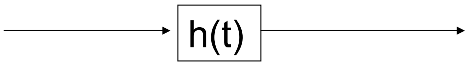
Response of c.t. LTI systems to complex exponential with magnitude one



$x(t) = e^{j\omega_0 t}$

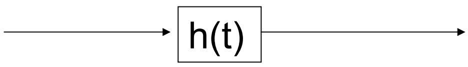
$\omega_0 \in R, t \in R$

$$y(t) = \int_{-\infty}^{\infty} h(\tau) e^{j\omega_0 (t-\tau)} d\tau$$

$$y(t) = e^{j\omega_0 t} \cdot \underbrace{\int_{-\infty}^{\infty} h(\tau) e^{-j\omega_0 \tau} d\tau}_{H(\omega_0)}$$

Fourier transform of *h*, computed in ω_0

depends on ω_0 and *h* 3



$x(t) = e^{j\omega_0 t}$

eigenfunction

$y(t) = e^{j\omega_0 t} \cdot H(\omega_0)$

eigenvalue

$$H(\omega) = \int_{-\infty}^{\infty} h(\tau) e^{-j\omega\tau} d\tau = |H(\omega)| e^{j\Phi(\omega)}$$

$$y(t) = e^{j\omega_0 t} H(\omega_0) = |H(\omega_0)| e^{j(\omega_0 t + \Phi(\omega_0))}$$

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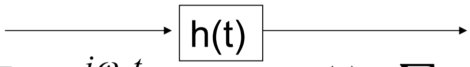

$$x(t) = \sum_k a_k e^{j\omega_k t} \qquad y(t) = \sum_k a_k H(\omega_k) e^{j\omega_k t}$$

- input = linear combination of complex exponentials $\Rightarrow$ the *output = a linear* combination of complex exponentials

$$y(t) = \sum_k a_k S\left\{e^{j\omega_k t}\right\} = \sum_k a_k H(\omega_k) e^{j\omega_k t}$$

$$\Downarrow$$

$$H(\omega_k) e^{j\omega_k t}$$

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Orthogonal Transforms

- Scalar product of two vectors

$$\mathbf{x} = \left[x_1 \ x_2 \ \dots \ x_n\right]^T; \quad \mathbf{y} = \left[y_1 \ y_2 \ \dots \ y_n\right]^T$$

$$\langle \mathbf{x}, \mathbf{y} \rangle = \left[x_1 \ x_2 \ \dots \ x_n\right] \begin{bmatrix} y_1^* \\ y_2^* \\ \dots \\ y_n^* \end{bmatrix} = x_1 y_1^* + x_2 y_2^* + \dots + x_n y_n^*$$

•Scalar product of functions from to $L^2{}_{[a,b]}$

$$\langle x(t), y(t) \rangle = \int_a^b x(t) y^*(t) \, dt$$

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- Properties

i) $\langle x, y\rangle = \langle y, x\rangle^*$,

ii) $\langle x, y+z\rangle = \langle x, y\rangle + \langle x, z\rangle$,

iii) $\langle \lambda x, y\rangle = \lambda\langle x, y\rangle$,

iv) $\langle x, \lambda y\rangle = \lambda^*\langle x, y\rangle \;\; \forall\lambda \in C$,

v) $\left\langle \sum_{k=1}^{n}\alpha_k x_k, \sum_{l=1}^{m}\beta_l y_l \right\rangle = \sum_{k=1}^{n}\sum_{l=1}^{m}\alpha_k\beta_l^*\langle x_k, y_l\rangle.$

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Proof

$$\left\langle \sum_{k=1}^{n}\alpha_k x_k, \sum_{l=1}^{m}\beta_l y_l \right\rangle \overset{ii)}{=} \sum_{l=1}^{m}\left\langle \sum_{k=1}^{n}\alpha_k x_k, \beta_l y_l \right\rangle =$$

$$\overset{iii)}{=} \sum_{l=1}^{m}\beta_l^*\left\langle \sum_{k=1}^{n}\alpha_k x_k, y_l \right\rangle \overset{i)}{=} \sum_{l=1}^{m}\beta_l^*\left\langle y_l, \sum_{k=1}^{n}\alpha_k x_k \right\rangle^* =$$

$$\overset{ii)}{=} \sum_{l=1}^{m}\sum_{k=1}^{n}\beta_l^*\langle y_l, \alpha_k x_k\rangle^* \overset{iii)}{=} \sum_{l=1}^{m}\sum_{k=1}^{n}\beta_l^*\alpha_k\langle y_l, x_k\rangle^* =$$

$$\overset{i)}{=} \sum_{k=1}^{n}\sum_{l=1}^{m}\alpha_k\beta_l^*\langle x_k, y_l\rangle$$

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- The norm

$$\|\mathbf{x}\|^2 = \langle \mathbf{x}, \mathbf{x} \rangle = |x_1|^2 + |x_1|^2 + \ldots + |x_1|^2 = \sum_{k=1}^{n} |x_k|^2$$

$$\|x\|^2 = \int_a^b |x(t)|^2 dt$$

- Rules i)-iv) apply for the space L^2, so the norms ||x|| are finite.

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Hilbert space

- A structure frequently used in approximation theory.
- Space composed by vectors. Each of them has its norm.
- This norm is defined with the aid of the scalar product of vectors denoted by <>.

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1) Finite dimensional Hilbert space, with dimension *n*

$$x = (x_1, x_2, ..., x_n)^T , y = (y_1, y_2, ..., y_n)^T ,$$

$$\langle x, y \rangle = x^T y^* = (x_1, x_2, ..., x_n) \begin{bmatrix} y_1^* \\ y_2^* \\ \cdot \\ \cdot \\ \cdot \\ y_n^* \end{bmatrix} = \sum_{k=1}^{n} x_k y_k^* ,$$

$$\|x\|^2 = \langle x, x \rangle = \sum_{k=1}^{n} |x_k|^2$$

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- The scalar product <x,y>-matrices product of the transposed of the x with the conjugate of the y.
- The squared norm of *x* = the scalar product <*x*,*x*>.
- Model for:
  - Discrete time signals on the interval [0, n-1]
  - or periodic (with period n) discrete time signals.

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2) Finite dimensional Hilbert space of finite energy signal with finite duration

$x, y \in L^2_{[a,b]}$;

$$\langle x(t), y(t)\rangle = \int_a^b x(t)y^*(t)\,dt; \|x(t)\|^2 = \int_a^b |x(t)|^2\,dt.$$

$$\left\{x : R \to C \left| \int_a^b |x(t)|^2\,dt < \infty \right.\right\}$$

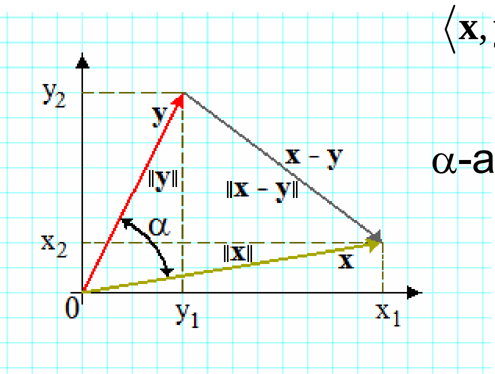
- Model for:
  - –continuous time signals on the interval [a,b] or
  - –periodic (with period T=b-a) continuous time signals

Orthogonal vectors

- For two bidimensional vectors

$$\mathbf{x} = \mathbf{i}x_1 + \mathbf{j}x_2 \;;\; \mathbf{y} = \mathbf{i}y_1 + \mathbf{j}y_2$$

$$\langle \mathbf{x}, \mathbf{y}\rangle = \|\mathbf{x}\|\|\mathbf{y}\|\cos\alpha$$



α-angle between vectors

$$\cos\alpha = \frac{\langle \mathbf{x}, \mathbf{y}\rangle}{\|\mathbf{x}\|\cdot\|\mathbf{y}\|}$$

- If the two vectors are perpendicular (orthogonal) then <**x**, **y**> = 0
- If the scalar product is 0 => the two vectors are orthogonal.
- **Orthogonality condition:**

$$\langle \mathbf{x}, \mathbf{y} \rangle = 0 \Leftrightarrow \mathbf{x} \perp \mathbf{y}$$

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Orthogonal functions

- Consider two signals defined on $(0,T_0)$, with $T_0=2\pi/\omega_0$ -- space $L^2_{[0,T0]}$

$$x(t) = \cos \omega_0 t \;;\; y(t) = \sin \omega_0 t$$

- The scalar product is 0

$$\langle \cos \omega_0 t, \sin \omega_0 t \rangle = \int_0^{T_0} \cos(\omega_0 t)\sin(\omega_0 t) dt = \frac{1}{2}\int_0^{T_0} \sin(2\omega_0 t)\, dt$$

$$= -\frac{\cos(2\omega_0 t)}{4\omega_0}\Bigg|_0^{T_0} = \frac{1-\cos 4\pi}{4\omega_0} = 0$$

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Complete space

- A system U={u_k} of orthogonal vectors two by two from a Hilbert space H is **complete** in H if:
- there is no other vector x∈H-U , orthogonal on all the vectors from U **only the vector 0**

$$\langle u_k, x \rangle = 0 \iff x = 0, \text{ if } x \in H - U.$$

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Orthogonal basis of Hilbert space

- Each complete system U is an orthogonal basis of H.
- Any element x from H can be expressed like a linear combination of elements of *U* **uniquely**

$$\forall x \in H,\ x = \sum_k a_k u_k.$$

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Examples of basis

- The **unity vectors** {**i, j, k**} are a **basis** in the three-dimensional space.

- The set of **complex exponential** $\{e^{jk\omega_0 t}\}|_{k\in Z}$ with frequency $k\omega_0$ is an infinite dimensional basis for the periodic continuous time signals of period T_0

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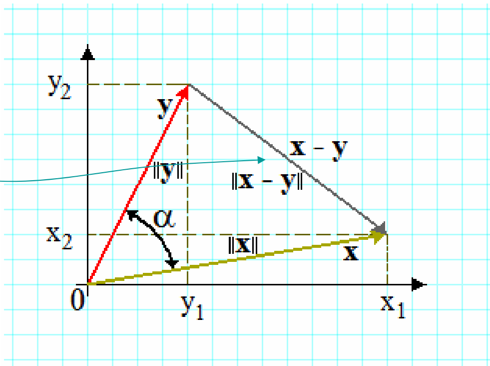
Pythagoras' theorem in the Hilbert space. Relation between distance and scalar product

- Consider two vectors in the Hilbert space
- Their difference is

$$\mathbf{d} = \mathbf{x} - \mathbf{y}$$

$$d^2(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|^2$$



- For vectors/functions in the Hilbert space

$$d^2(x,y)=\|x-y\|^2=\|x\|^2-2\operatorname{Re}\{\langle x,y\rangle\}+\|y\|^2$$

- **Pythagoras' theorem in the Hilbert space.** If *x* and *y* are orthogonal

$$d^2(x,y)=\|x\|^2+\|y\|^2$$

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Examples for Pythagoras' theorem in the Hilbert space $L^2_{[0,T0]}$

- orthogonal signals $\cos(\omega_0 t)$ and $\sin(\omega_0 t)$ have the same norm

$$\|x(t)\|^2=\int_0^{T_0}\cos^2(\omega_0 t)\,dt=\int_0^{T_0}\frac{1+\cos(2\omega_0 t)}{2}dt=$$

$$=\frac{t}{2}\Big|_0^{T_0}+\frac{1}{2}\cdot\frac{1}{2\omega_0}\sin(2\omega_0 t)\Big|_0^{T_0}=\frac{T_0}{2}$$

- Pythagoras' theorem

$$d^2(\cos\omega_0 t,\sin\omega_0 t)=\frac{T_0}{2}+\frac{T_0}{2}=T_0.$$

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- non-orthogonal signals do not satisfy Pythagoras' theorem.
- signals on $L^2_{[0,T0]}$ $\cos(\omega_0 t)$ and $-\cos(\omega_0 t)$

are not orthogonal:

$$\langle \cos \omega_0 t, -\cos \omega_0 t \rangle = -\int_0^{T_0} \cos^2 \omega_0 t dt = -\frac{T_0}{2}.$$

$$d^2(x,y) = \|x\|^2 - 2\operatorname{Re}\{\langle x,y \rangle\} + \|y\|^2$$

Should be zero, but it's not

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Examples for Pithagoras' theorem

- square distance

$$d^2(x,y) = \|x\|^2 - 2\operatorname{Re}\{\langle x,y \rangle\} + \|y\|^2$$

$$\|x\|^2 = \|y\|^2 = T_0/2$$

- scalar product $\langle x,y \rangle = -T_0/2$.
- $\Rightarrow$ the square distance

$$d^2(\cos \omega_0 t, -\cos \omega_0 t) = \frac{T_0}{2} + \frac{T_0}{2} - 2\left(-\frac{T_0}{2}\right) = 2T_0.$$

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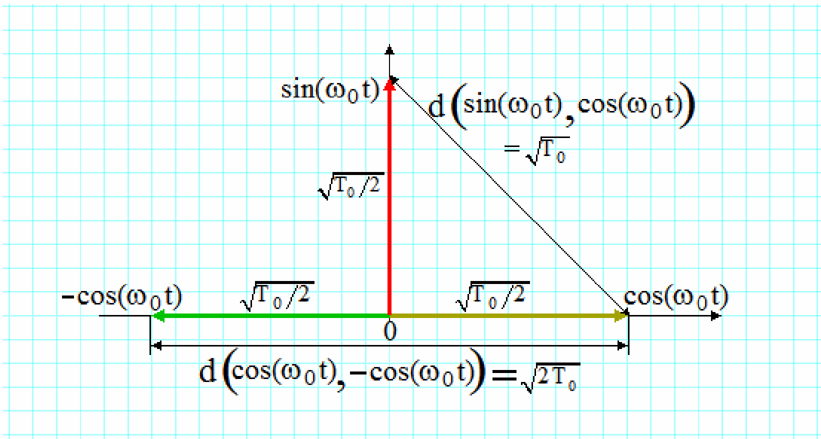

$$d\left(\cos\omega_0 t, \sin\omega_0 t\right) < d\left(\cos\omega_0 t, -\cos\omega_0 t\right).$$

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Schwarz's inequality in the Hilbert space

(Cauchy- Bunyakovsky-Schwarz inequality)

$$\left|\langle x, y\rangle\right| \le \|x\| \cdot \|y\|$$

- with equality holding *if and only if x* and *y* are linearly dependent, i.e.

$$y=kx,$$

- for some scalar k

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Examples for Schwarz's inequality 1. Orthogonal signals $L^2_{[0,T_0]}$

$$x(t)=\cos(\omega_0 t) \qquad y(t)=\sin(\omega_0 t)$$

$$\left|\langle x(t), y(t)\rangle\right| = 0$$

- product of the norms

$$\|x(t)\|\cdot\|y(t)\| = \sqrt{\frac{T_0}{2}}\cdot\sqrt{\frac{T_0}{2}} = \frac{T_0}{2}$$

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- Schwarz's inequality is verified

$0<T_0/2$

- There is <u>no k</u> such that y(t) = k·x(t)
- So, in this case the Schwarz's inequality can not become equality

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Examples for Schwarz's inequality 2. Non-orthogonal signals $L^2_{[0,T_0]}$

$$x(t)=\cos(\omega_0 t) \quad \text{and} \quad y(t)=-\cos(\omega_0 t)$$

$$\Rightarrow y(t)=-x(t)$$

- there exists a value *k=-1* - Schwarz's inequality becomes an equality.

$$\|x(t)\|=\sqrt{\frac{T_0}{2}}; \quad \|y(t)\|=\sqrt{\frac{T_0}{2}}$$

$$\left|\langle x(t), y(t)\rangle\right|=\left|-\frac{T_0}{2}\right|=\frac{T_0}{2}$$

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Optimal approximation in Hilbert space

- n-dimensional Hilbert space, with orthogonal basis $U=\{u_1,u_2,...,u_n\}$

$$\langle u_k,u_l\rangle=\begin{cases}\|u_l\|^2, & k=l\\ 0, & k\neq l\end{cases}$$

- U is orthonormal

$$\|u_k\|^2=1 \text{ and } c_k=\langle x,u_k\rangle.$$

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- Vector=unique linear combination of vectors from U

$$x = \sum_{k=1}^{n} c_k u_k$$

- The coefficients c_k

$$c_k = \frac{\langle x, u_k \rangle}{\|u_k\|^2}, \quad k \in \{1,\ 2,\ ...,n\}\ ,\ \forall x \in H.$$

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Optimal approximation in Hilbert space

- Approximation: represent *n*-dimensional vector x using only *m* elements, *m*<*n*

$$\tilde{x} = \sum_{k=1}^{m} \lambda_k u_k$$

- **Best approximation: truncation of its series decomposition**

$$\boxed{\lambda_k = c_k, k = 1,...,m.}$$

- increase *m* (number of terms in the approximation)

$\Rightarrow$ the error decreases & the approximation becomes better

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Proof

- approximation error

$$e = x - \tilde{x}$$

- norm

$$d^2(x, \tilde{x}) = \|x - \tilde{x}\|^2 = \|e\|^2$$

- minimize the norm of the error

$$\|e\| = d(x, \tilde{x})$$

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$$d^2(x, \tilde{x}) = \|e\|^2 = \langle e, e \rangle = \left\langle x - \sum_{k=1}^{m} \lambda_k u_k, x - \sum_{i=1}^{m} \lambda_i u_i \right\rangle$$

$$= \|x\|^2 - \sum_{k=1}^{m} \lambda_k \langle u_k, x \rangle - \sum_{i=1}^{m} \lambda_i^* \langle x, u_i \rangle + \sum_{k=1}^{m} \sum_{i=1}^{m} \lambda_k \lambda_i^* \langle u_k, u_i \rangle$$

$$= \|x\|^2 - \sum_{k=1}^{m} \left(\lambda_k \langle x, u_k \rangle^* + \lambda_k^* \langle x, u_k \rangle \right) + \sum_{k=1}^{m} |\lambda_k|^2 \|u_k\|^2$$

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- Select coefficients λ_k to minimize d^2.
- Partial derivates of d^2 (function of λ_k) = 0

$$\lambda_k = \frac{\langle x, u_k \rangle}{\|u_k\|^2} = c_k,\ k \in \{1, 2, \dots, m\}\ ,\ m < n,$$

$$d_{\min}^2(x, \tilde{x}) = \|x\|^2 - \sum_{k=1}^{m} \frac{|\langle x, u_k \rangle|^2}{\|u_k\|^2} = \|x\|^2 - \sum_{k=1}^{m} |c_k|^2 \|u_k\|^2.$$

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Projection theorem

$$d_{\min}^2(x, \tilde{x}) = \|x\|^2 - \|\tilde{x}\|^2$$

$$\|x - \tilde{x}\|^2 = \|x\|^2 - \|\tilde{x}\|^2$$

$$\Rightarrow \|x\|^2 = \|\tilde{x}\|^2 + \|x - \tilde{x}\|^2$$

- the approximation error is orthogonal on $\tilde{x}$ so it's orthogonal on the approximation *m*-dimensional subspace

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- For H a Hilbert space, H_s a closed subspace of H
- For each vector x in H there is a vector $\tilde{x}$ in H_s = the best approximation of x with elements in H_s with the properties
  1. The distance from x to $\tilde{x}$ is smaller than the distance from x to each element from H_s
  2. The error produced $e = x - \tilde{x}$ is orthogonal on the subspace H_s

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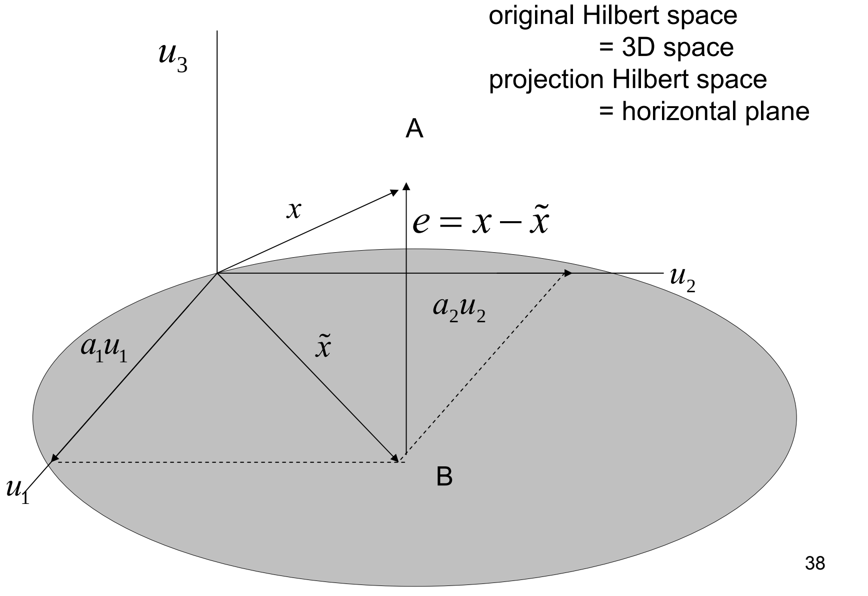
$$x = O\overline{A}, \tilde{x} = O\overline{B}, e = B\overline{A}$$



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$$\|x\|^2 = \langle x, x \rangle = \left\langle \sum_{k=1}^{n} c_k u_k , \sum_{l=1}^{n} c_l u_l \right\rangle = \sum_{k=1}^{n} \sum_{l=1}^{n} c_k c_l \langle u_k , u_l \rangle$$

$$= \sum_{k=1}^{n} c_k^2 \|u_k\|^2$$

$$d_{\min}^2 (x, \tilde{x}) = \|e\|_{\min}^2 = \|x\|^2 - \|\tilde{x}\|^2 =$$

$$= \sum_{k=1}^{n} |c_k|^2 \|u_k\|^2 - \sum_{k=1}^{m} |c_k|^2 \|u_k\|^2 = \sum_{k=m+1}^{n} |c_k|^2 \|u_k\|^2$$

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Infinite dimensional Hilbert spaces

- $U_N = \{u_k(t),\ k = -N, N\}$ orthogonal basis in a finite dimensional space, subspace of Hilbert space
- The decomposition of signal x:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k u_k(t), \ \text{ with } c_k = \frac{\langle x(t), u_k(t) \rangle}{\|u_k(t)\|^2}$$

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The case of infinite dimensional spaces

- Approximation signal in a finite dimensional Hilbert space of dimension $2N+1$:

$$\tilde{x}_N(t) = \sum_{k=-N}^{N} \lambda_k u_k(t)$$

- Like before, we have

$$\boxed{\lambda_k = c_k, \quad k \in \{-N, -N+1, \ldots, 0, 1, \ldots, N\}}$$

- for minimum error

$$\left\| x(t) - \tilde{x}_N(t) \right\|^2 = \left\| x(t) \right\|^2 - \sum_{k=-N}^{N} |c_k|^2 \left\| u_k(t) \right\|^2$$

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- But:

$$\left\| x(t) \right\|^2 = \langle x(t), x(t) \rangle = \left\langle \sum_{k=-\infty}^{\infty} c_k u_k(t), \sum_{l=-\infty}^{\infty} c_l u_l(t) \right\rangle$$

$$= \sum_{k=-\infty}^{\infty} |c_k|^2 \left\| u_k(t) \right\|^2$$

Parseval's relation

- The error becomes:

$$\left\| x(t) - \tilde{x}_N(t) \right\|^2 = \sum_{k=-\infty}^{\infty} |c_k|^2 \left\| u_k(t) \right\|^2 - \sum_{k=-N}^{N} |c_k|^2 \left\| u_k(t) \right\|^2$$

$$= \sum_{\forall |k| > N} |c_k|^2 \left\| u_k(t) \right\|^2$$

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The case of infinite dimensional spaces

- More terms (N high) $\Rightarrow$ error decreases
- We have :

$$\left\|\tilde{x}_N(t)\right\|^2 = \sum_{k=-N}^{N} \left|c_k\right|^2 \left\|u_k\right\|^2 \leq \left\|x(t)\right\|^2$$

- **Bessel's inequality**.

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$$\left\|x(t)\right\|^2 < \infty \text{ because } x(t) \in L^2_{[a,b]}$$

$$\Rightarrow \lim_{N\to\infty} \sum_{\forall|k|>N} \left|c_k\right|^2 \left\|u_k\right\|^2 = 0$$

$$\Rightarrow \lim_{N\to\infty} \left\|x(t) - \tilde{x}_N(t)\right\|^2 = 0$$

- The approximation signal $\tilde{x}_N(t)$ converges **in mean square** to *x(t)*

$$\underset{N\to\infty}{\text{l.i.m.}}\, \tilde{x}_N(t) = x(t)$$

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Remarks

1. We have $\|x(t)\|^2 = \|\tilde{x}_N(t)\|^2 + \|x(t) - \tilde{x}_N(t)\|^2$

Pitagora's theorem: orthogonality between the best approximation and the approximation error

$$\langle \tilde{x}_N(t), x(t) - \tilde{x}_N(t) \rangle = 0$$

2. **Parseval**'s relation (**Rayleigh's energy** theorem)

$$W = \|x(t)\|^2 = \sum_{k=-\infty}^{\infty} |c_k|^2 \|u_k(t)\|^2$$

3. The best approximation is obtained by truncating the series decomposition

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Fourier Series

- consider in the space $L^2_{[0,T_0]}$ an orthogonal basis :

$$u_k(t) = e^{jk\omega_0 t},\ k \in Z$$

- The elements are orthogonal and the set is complete.

$$\langle e^{jk\omega_0 t}, e^{jl\omega_0 t} \rangle = \int_0^{T_0} e^{j(k-l)\omega_0 t} = \begin{cases} 0, k \neq l \\ T_0, k = l \end{cases}$$

The norm $\|u_k(t)\|^2 = T_0$

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Exponential Fourier series

- For a periodic signal x(t)=x(t+T_0)

$$x(t) = \sum_{k=-\infty}^{\infty} c_k u_k(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

$$c_k = \frac{\langle x(t), e^{jk\omega_0 t} \rangle}{\left\| e^{jk\omega_0 t} \right\|^2} = \frac{1}{T_0} \int_{T_0} x(t) e^{-jk\omega_0 t} dt$$

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \leftrightarrow c_k = \frac{1}{T_0} \int_{T_0} x(t) e^{-jk\omega_0 t} dt, \quad \omega_0 = \frac{2\pi}{T_0}$$

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Trigonometric Fourier series

- Euler's relations

$$\cos(k\omega_0 t) = \frac{1}{2}\left(e^{jk\omega_0 t} + e^{-jk\omega_0 t}\right); \ \sin(k\omega_0 t) = \frac{1}{2j}\left(e^{jk\omega_0 t} - e^{-jk\omega_0 t}\right)$$

- An orthogonal basis of the same space is:

$$U = \{1, \ \cos(k\omega_0 t), \sin(k\omega_0 t)\}_{k \in N}$$

- the elements are orthogonal and the set is complete.

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Trigonometric Fourier series

- The norms of the basis' elements are:

$$\|1\|^2 = \int_{-T_0/2}^{T_0/2} 1^2\,dt = T_0;$$

$$\|\cos(k\omega_0 t)\|^2 = \int_{-T_0/2}^{T_0/2} \cos^2(k\omega_0 t)\,dt = \frac{T_0}{2};$$

$$\|\sin(k\omega_0 t)\|^2 = \int_{-T_0/2}^{T_0/2} \sin^2(k\omega_0 t)\,dt = \frac{T_0}{2};$$

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Trigonometric Fourier series

- So, any periodic signal of period T_0 can be expressed in the form:

$$x(t) = a_0 \cdot 1 + \sum_{k=1}^{\infty}\left(a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t)\right)$$

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Trigonometric Fourier series

- the coefficients are :

$$a_0 = \frac{\langle x(t),1\rangle}{\|1\|^2} = \frac{1}{T_0}\int_{T_0} x(t)dt, \text{ continuous component.}$$

$$a_k = \frac{\langle x(t),\cos(k\omega_0 t)\rangle}{\|\cos(k\omega_0 t)\|^2} = \frac{2}{T_0}\int_{T_0} x(t)\cos(k\omega_0 t)\,dt,$$

$$b_k = \frac{\langle x(t),\sin(k\omega_0 t)\rangle}{\|\sin(k\omega_0 t)\|^2} = \frac{2}{T_0}\int_{T_0} x(t)\sin(k\omega_0 t)\,dt.$$

Remarks

1. a_0 - DC component of the signal $x(t)$
2. The signal with no DC component(a_0 =0) has only “oscillatory” components:

$$x(t) = \sum_{k=1}^{\infty}\left(a_k \cos k\omega_0 t + b_k \sin k\omega_0 t\right);$$

$$x(t) - \text{odd} \Rightarrow a_k = 0; \quad x(t) - \text{even} \Rightarrow b_k = 0;$$

3. For real signals

$$x(t) = x^*(t) \Rightarrow c_{-k} = c_k^*$$

$$c_{-k} = \frac{1}{T_0}\int_{T_0} x(t)e^{jk\omega_0 t}dt = \left[\frac{1}{T_0}\int_{T_0} x(t)e^{-jk\omega_0 t}dt\right]^* = c_k^*$$

4. the power of the signal $x(t)$ - Parseval's relation :

$$P = \frac{W}{T_0} = \sum_{k=1}^{\infty} |c_k|^2 \frac{T_0}{T_0} \Rightarrow P = \sum_{k=1}^{\infty} |c_k|^2 = \frac{1}{T_0} \int_{T_0} |x(t)|^2$$

- another form of the Parseval's relation:

$$P = \frac{1}{T_0} \int_{T_0} |x(t)|^2 \, dt = a_0^2 + \sum_{k=1}^{\infty} \left(\frac{a_k^2}{2} + \frac{b_k^2}{2} \right)$$

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Harmonic Fourier Series

- Using the relation:

$$a_k \cos k\omega_0 t + b_k \sin k\omega_0 t = \sqrt{a_k^2 + b_k^2} \cos(k\omega_0 t + \varphi_k)$$

$$\mathrm{tg}\, \varphi_k = -\frac{b_k}{a_k}. \; A_k = \sqrt{a_k^2 + b_k^2}$$

- the Fourier trigonometric series becomes:

$$x(t) = \sum_{k=0}^{\infty} A_k \cos(k\omega_0 t + \varphi_k)$$

- **harmonic form**.

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Relations between coefficients

- For real signals we have

$$|c_k| = \sqrt{a_k^2 + b_k^2} = \frac{1}{2} A_k, \; k \geq 1$$

$$|c_k| = |c_{-k}| \;, \; k \leq -1;$$

$$\arg c_k = \varphi_k \;, \; k \geq 1 \;;$$

$$\arg c_{-k} = -\varphi_k \;, \; k \leq -1;$$

$$|c_0| = |a_0| \;; \; \arg c_0 = 0.$$

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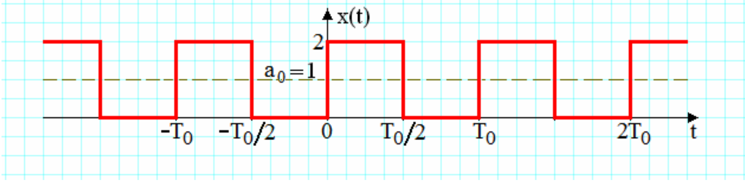
Spectrum diagrams

- represent periodic signals in the frequency domain.

$$x(t) = \begin{cases} 2, & 0 \leq t < \frac{T_0}{2} \\ 0, & \frac{T_0}{2} \leq t < T_0 \end{cases}$$



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- DC component:

$$a_0 = \frac{1}{T_0} \int_{T_0} x(t) dt = \frac{1}{T_0} \int_0^{T_0/2} 2dt = 1; \quad A_0 = a_0$$

- The oscillatory component is odd

$$x_1(t) = \begin{cases} 1, & 0 \le t < \frac{T_0}{2} \\ -1, & \frac{T_0}{2} \le t < T_0 \end{cases} \qquad k \neq 0 \Rightarrow a_k = 0$$

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- the other coefficients

$$b_k = \frac{2}{T_0} \int_{T_0} x(t) \sin k\omega_0 t dt = \frac{4}{T_0} \cdot \left. \frac{-\cos k\omega_0 t}{k\omega_0} \right|_0^{T_0/2} = \frac{4}{T_0} \cdot \frac{1-(-1)^k}{k\omega_0}; \quad k \ge 1$$

- or

$$b_{2k-1} = \frac{4}{(2k-1)\pi}; \quad k = 1, 2, 3, \ldots$$

$$b_{2k} = 0$$

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$$x(t) = 1 + \sum_{k=1}^{\infty} \frac{4}{(2k-1)\pi} \sin\left[(2k-1)\omega_0 t\right]$$

$$x(t) = 1 + \sum_{k=1}^{\infty} \frac{4}{(2k-1)\pi} \cos\left[(2k-1)\omega_0 t - \frac{\pi}{2}\right]$$

A_{2k-1},

$(2k-1)$ th order harmonics of frequency $(2k-1)\omega_0$

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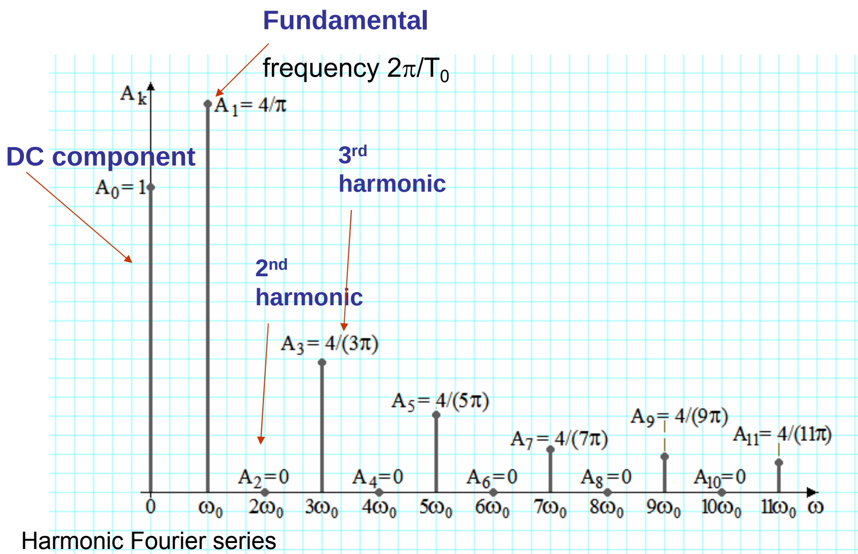
Amplitude spectrum ($k\omega_0$, A_k)



Harmonic Fourier series

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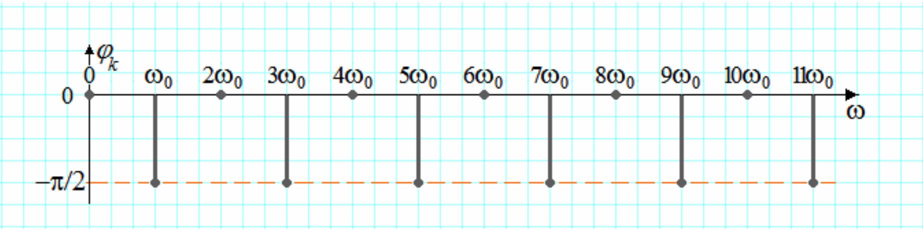
Phase spectrum ($k\omega_0$, φ_k)



Harmonic Fourier series

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Amplitude spectrum ($k\omega_0$, $|c_k|$)

- obtained also with the **complex exponential form** of the Fourier series.
- The coefficients c_k :

$$c_0 = \frac{1}{T_0}\int_{T_0} x(t)\,dt = a_0 = 1$$

$$c_k = \frac{1}{T_0}\int_{T_0} x(t)e^{-jk\omega_0 t}dt = \frac{1}{T_0}\int_0^{T_0/2} 2e^{-jk\omega_0 t}dt = \frac{1-(-1)^k}{jk\pi}; \quad k \neq 0$$

$$c_{2k-1} = \frac{2}{|2k-1|\pi}e^{j\frac{\pi}{2}}; \quad k \leq -1; \quad c_{2k-1} = \frac{2}{(2k-1)\pi}e^{-j\frac{\pi}{2}}; \quad k \geq 1$$

$$c_{2k} = 0, \quad k \neq 0$$

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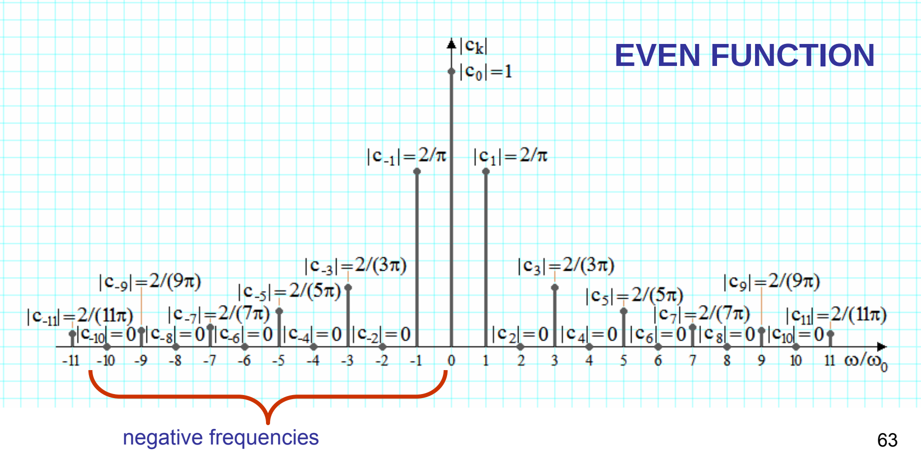
Magnitude spectrum ($k\omega_0$, $|c_k|$)

$$|c_{2k-1}| = \frac{2}{|2k-1|\pi}$$



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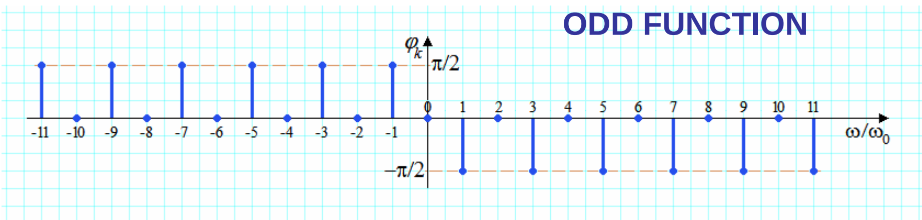
Phase spectrum ($k\omega_0$, φ_k)

- Another representation in the frequency domain. For the square wave we have:

$$\varphi_k = -\frac{\pi}{2}\operatorname{sgn}(k)$$



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Other forms of Parseval's relation

- complex exponential Fourier series :

$$P = \frac{1}{T_0}\int_{T_0} |x(t)|^2\,dt = \sum_{k=-\infty}^{\infty} |c_k|^2 = |c_0|^2 + 2\sum_{k=0}^{\infty} |c_k|^2$$

- trigonometric & harmonic

$$P = a_0^2 + \sum_{k=1}^{\infty}\left(\frac{a_k^2}{2} + \frac{b_k^2}{2}\right) = \frac{1}{T_0}\int_{T_0} |x(t)|^2\,dt = A_0^2 + \sum_{k=1}^{\infty}\frac{A_k^2}{2}$$

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- The power of this square wave

$$P = \frac{1}{T_0}\int_{T_0} |x(t)|^2\,dt = \frac{1}{4}\int_{0}^{T_0/2} 4dt = 2$$

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Power spectrum

- the association of the frequencies of its components with their powers
  - harmonic form ($k\omega_0$, $A_k^2/2$)
  - complex exponential form ($k\omega_0$, $|c_k|^2$)

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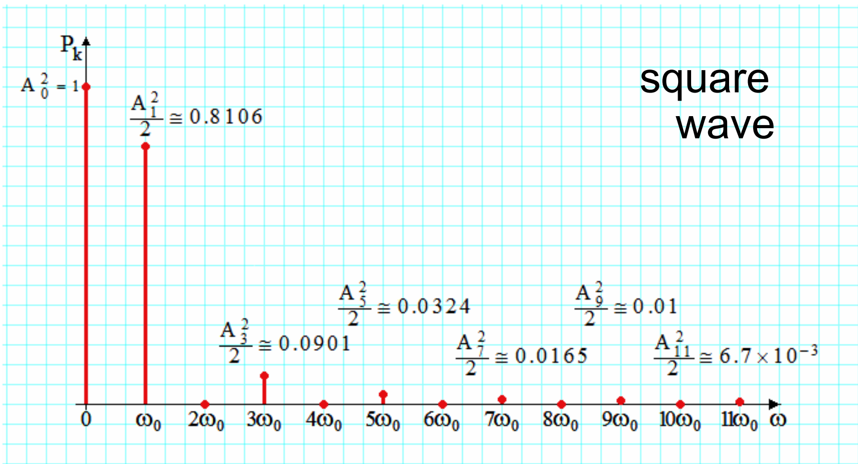
Power spectrum with the harmonic Fourier series



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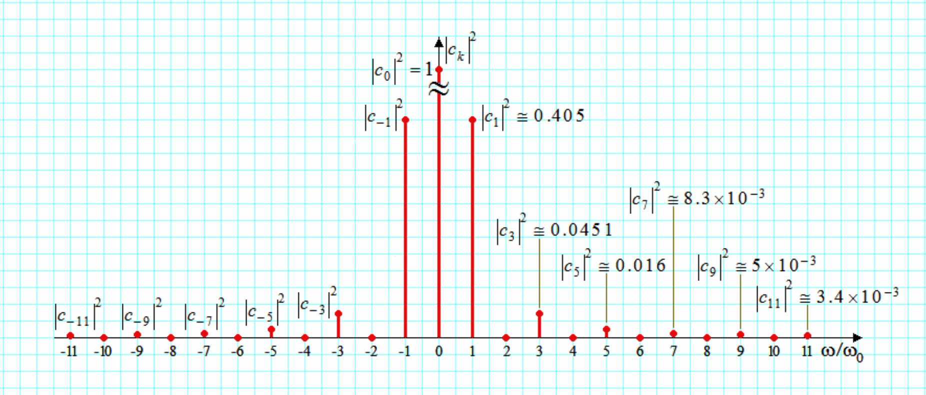
Power spectrum with the exponential Fourier series



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- non-band-limited signal:
  - the signal has infinite frequency bandwidth.
  - The power decreases as the frequency increases; it approaches zero only at infinite frequency
- ***effective bandwidth*** of the signal = positive frequency range with a “significant” percentage of the power of the signal.
- For this case, in the bandwidth $9\omega_0$ we find 96,5% of the power of the signal

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Gibbs' Phenomenon

- The physicist **Albert Michelson** tried to construct a spectrum analyzer in 1898.
- He observed that the spectrum analyzer was not working properly for non band-limited signals.
- He asked to **Gibbs** to explain this phenomenon.

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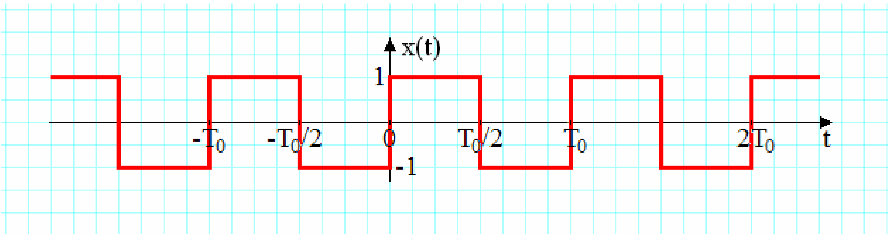
Gibbs considered the following non band-limited signal:



a square wave with duty factor 0.5 with no DC component

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- Fourier expansion, non band-limited signal

$$x(t)=\frac{4}{\pi}\left[\sin\omega_0 t+\frac{1}{3}\sin 3\omega_0 t+\frac{1}{5}\sin 5\omega_0 t+\ldots\right]$$

- truncation in frequency : non band-limited input signal was approximated with a band-limited signal, *n* odd harmonics

$$\tilde{x}(t)=\frac{4}{\pi}\left[\sin\omega_0 t+\frac{1}{3}\sin 3\omega_0 t+\frac{1}{5}\sin 5\omega_0 t+\ldots+\frac{1}{2n-1}\sin(2n-1)\omega_0 t\right]$$

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$$\tilde{x}(t)=\frac{2}{\pi}\int_0^{2n\omega_0 t}\frac{\sin u}{u}du=\frac{2}{\pi}\mathrm{Si}(2n\omega_0 t)$$

- Si(x) – sine integral – odd function

$$\mathrm{Si}(x)=\int_0^x\frac{\sin u}{u}du;\quad \mathrm{Si}(-x)=-\mathrm{Si}(x)$$

$$\lim_{x\to\infty}\mathrm{Si}(x)=\frac{\pi}{2}$$

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Proof

$$\tilde{x}(t)=\frac{4\omega_0}{\pi}\int_0^t\left[\cos\omega_0\tau+\cos 3\omega_0\tau+\cos 5\omega_0\tau+...+\cos(2n-1)\omega_0\tau\right]d\tau$$

$$\cos\alpha+\cos(\alpha+r)+...+\cos\left[\alpha+(n-1)r\right]=\frac{\sin\frac{nr}{2}}{\sin\frac{r}{2}}\cos\left(\alpha+\frac{n-1}{2}r\right)$$

$\alpha=\omega_0\tau$ and $r=2\alpha$

$$\cos\omega_0\tau+\cos(3\omega_0\tau)+...+\cos\left[(2n-1)\omega_0\tau\right]$$
$$=\frac{\sin(n\omega_0\tau)}{\sin\omega_0\tau}\cos(n\omega_0\tau)=\frac{\sin(2n\omega_0\tau)}{2\sin\omega_0\tau}$$

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- The truncated Fourier series

$$\tilde{y}(t)=\frac{4\omega_0}{\pi}\int_0^t\left[\cos\omega_0\tau+\cos 3\omega_0\tau+...+\cos(2n-1)\omega_0\tau\right]d\tau=$$

$$\cong\frac{2}{\pi}\int_0^t\frac{1}{\tau}\sin\left(2n\cdot 2\pi\frac{\tau}{T_0}\right)d\tau$$

- approximated sin *x* = *x* (very small *x*).

$$0<\tau<t\ll T_0\Rightarrow\sin\left(2\pi\frac{\tau}{T_0}\right)\cong 2\pi\frac{\tau}{T_0}$$

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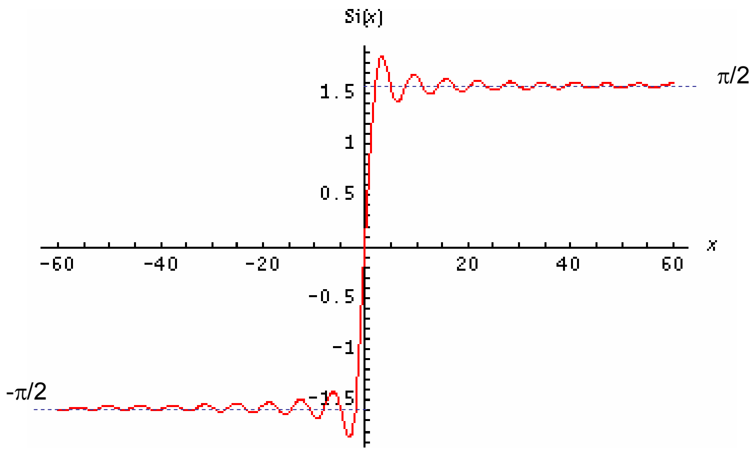
http://mathworld.wolfram.com/SineIntegral.html



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Gibbs' Phenomenon

- Gibbs proved that
  - truncating the square wave y(t) duty factor 0.5
  - preserving only *n* odd harmonic components

$$\tilde{x}(t) = \frac{4}{\pi}\left[\sin\omega_0 t + \frac{1}{3}\sin 3\omega_0 t + \frac{1}{5}\sin 5\omega_0 t + \ldots + \frac{1}{2n-1}\sin(2n-1)\omega_0 t\right]$$

- We have

$$\tilde{x}(t) = \frac{2}{\pi}\int_0^{2n\omega_0 t}\frac{\sin u}{u}\,du = \frac{2}{\pi}\mathrm{Si}(2n\omega_0 t)$$

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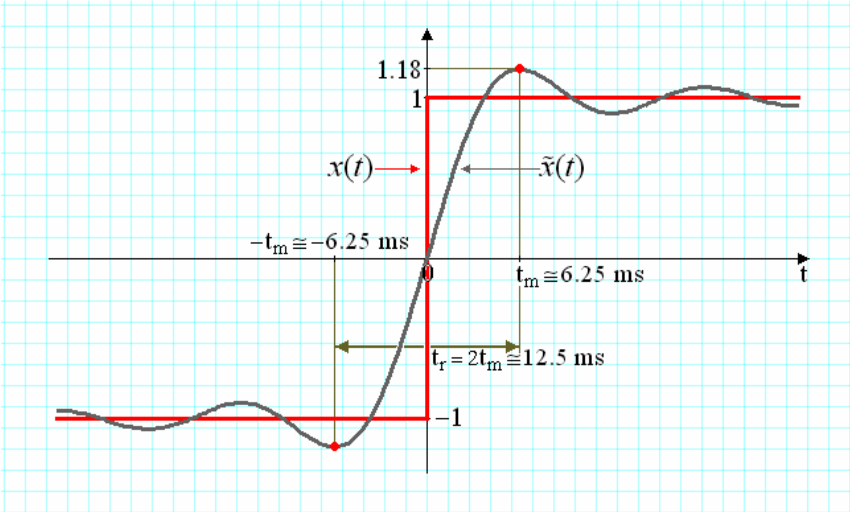
Gibbs phenomenon for a square wave, with T_0=1s(duty factor 0.5)



Gibbs' Phenomenon

- The approximation error is high in the neighborhood of the discontinuity.
- It has a damped oscillatory waveform.
- The maximum of the oscillation : 1.18 V , appears at the moment t_m.
- The rise time.

$$t_r \cong 2t_m = \frac{2\pi}{\omega_M} = \frac{1}{f_M}$$

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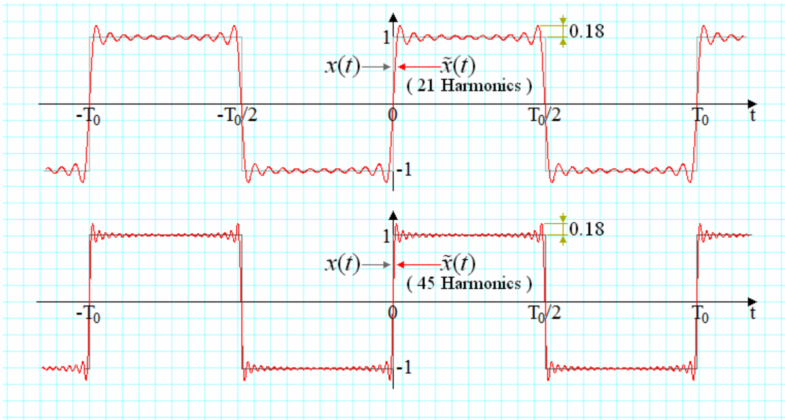
Truncated signals for 21 and 45 harmonics, respectively



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- maximum amplitude of the oscillations does not decrease
- the oscillation is compressed in time (its frequency increases).
- convergence in *mean square*.
- Gibbs' phenomenon proves that the non band-limited signals can't be perfectly approximated with band-limited signals

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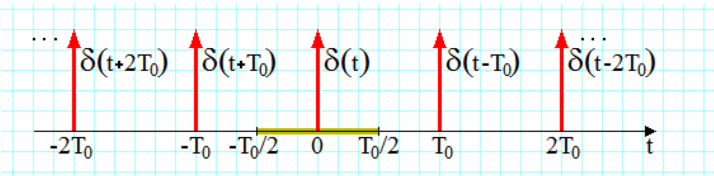
Periodic distributions

- Example: the Dirac periodic distribution, period T, $\delta_T(t)$

$$\delta_T(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT) \longrightarrow c_k = \frac{1}{T_0}$$



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$$c_k = \frac{1}{T} \int_{-T/2}^{T/2} \delta_T(t) e^{-jk\frac{2\pi}{T}t} dt = \frac{1}{T} \int_{-T/2}^{T/2} \delta(t) dt = \frac{1}{T}$$

$$\delta_T(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT) = \frac{1}{T} \sum_{k=-\infty}^{\infty} e^{jk\omega_0 t}$$

- for [-*T*/2,*T*/2] , $\delta_T(t) = \delta(t)$.
- The product of a c.t. function with δ(t)

$$x(t) \cdot \delta(t) = x(0) \cdot \delta(t)$$

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Exponential Fourier Series Properties

- Fourier coefficients of signal *x*, period *T*

$$x(t) \xleftrightarrow{\mathcal{F}} \{c_k^x\}$$

$$c_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt$$

- Fourier decomposition a.e. :

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \quad \text{a.e.w.}$$

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1. Linearity

- If the signals *x*(*t*) and *y*(*t*) are periodic with period *T* :

$$x(t) \xleftrightarrow{\mathcal{F}} \{c_k^x\} \quad , \quad y(t) \xleftrightarrow{\mathcal{F}} \{c_k^y\}$$

$$ax(t) + by(t) \xleftrightarrow{\mathcal{F}} \{ac_k^x + bc_k^y\}$$

- the Fourier decomposition - linear.

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2. Time shifting

- Time shifting $\rightarrow$ modulation with complex exponential.

$$x(t-t_0) \xleftrightarrow{\mathcal{F}} \left\{ e^{-jk\omega_0 t_0} c_k^x \right\}$$

$$c_k' = \frac{1}{T}\int_T x(t-t_0)\, e^{-jk\omega_0 t}\, dt = \frac{1}{T}\int_T x(\tau)\, e^{-jk\omega_0(\tau+t_0)}\, d\tau = e^{-jk\omega_0 t_0} c_k^x$$

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3. Complex conjugation

- Complex conjugation $\rightarrow$ reversal in frequency

$$x^*(t) \leftrightarrow c_{-k}^x$$

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4. Time reversal

- Time reversal $\rightarrow$ reversal in frequency.

$$c'_k = \frac{1}{T}\int_T x(-t)\,e^{-jk\omega_0 t}\,dt = \frac{1}{T}\int_T x(\tau)\,e^{-j(-k)\omega_0\tau}\,d\tau = c^x_{-k}$$

$$\breve{x}(t) = x(-t) \overset{\mathcal{F}}{\longleftrightarrow} \{c^x_{-k}\}$$

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5. Time scaling

- $x(t)$ - period $T \Rightarrow x(at)$, period $T/|a|$.
- The time scaled version has the same Fourier coefficients like the initial version.

$$c'_k = \frac{1}{T}\int_{T/a} x(at)\,e^{-jk\omega'_0 t}\,dt; \qquad \omega'_0 = \frac{2\pi}{\frac{T}{a}} = a\omega_0$$

$$c'_k = \frac{1}{T}\int_T x(\tau)\,e^{-jk\omega_0\tau}\,d\tau = c^x_k$$

$$x(at) \overset{\mathcal{F}}{\longleftrightarrow} \{c^x_k\}$$

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6. Signal's Modulation

- Modulation in time $\rightarrow$ frequency shifting

$$c'_k = \frac{1}{T}\int_T x(t)e^{jk_0\omega_0 t}e^{-jk\omega_0 t}dt = \frac{1}{T}\int_T x(t)e^{-j(k-k_0)\omega_0 t}dt = c^x_{k-k_0}$$

$$x(t)e^{jk_0\omega_0 t} \xleftrightarrow{\mathcal{F}} \left\{c^x_{k-k_0}\right\}$$

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Time-frequency duality

- operation in time $\rightarrow$ another operation in frequency :
  - modulation $\rightarrow$ shifting
- 2nd operation in time $\rightarrow$ first operation in frequency.
  - time shifting $\rightarrow$ modulation
- This behavior is named **duality**.
- Reversal is an auto-dual operation

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7. Product of signals

- discrete convolution of the Fourier coefficients sequences.

$$x(t)\,y(t) \xleftrightarrow{\mathcal{F}} \left\{ \sum_{n=-\infty}^{\infty} c_{k-n}^{x} c_{n}^{y} \right\} = \left\{ c_k^x * c_k^y \right\}$$

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8. Periodic convolution

- periodic signals do not finite energy their convolution can not be defined.
- circular convolution - for periodic signals.

$$z(t) = \int_T x(\tau)\,y(t-\tau)\,d\tau = x(t) \circledast y(t) \xleftrightarrow{\mathcal{F}} \left\{ T c_k^x c_k^y \right\}$$

- dual operations: multiplication ↔ convolution

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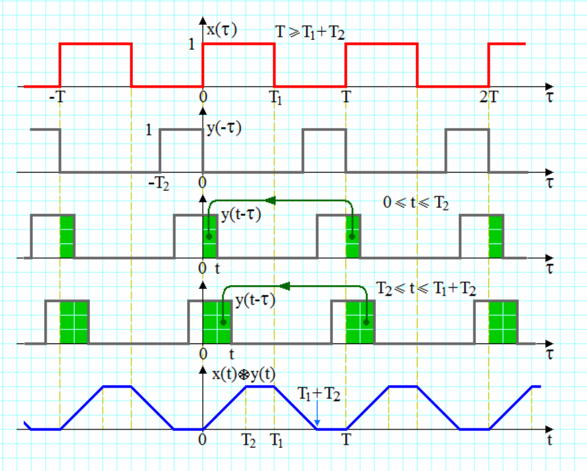
Circular convolution for 2 square waves, different duty factors

- The circularity effect can be observed.



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9. Signal Differentiation

- After differentiation, DC component =0.
- Time differentiation → multiplication with $jk\omega_0$.

$$\frac{dx(t)}{dt} \xleftrightarrow{\mathcal{F}} \left\{ jk\omega_0 c_k^x \right\}$$

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10. Signal's Integration

- Periodic signal with no DC component
- Time integration → multiplication with $1/jk\omega_0$.

$$\int_{-\infty}^{t} x(\tau)\,d\tau \xleftrightarrow{\mathcal{F}} \left\{\frac{c_k^x}{jk\omega_0}\right\} \qquad c_0^x = 0$$

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11. Real Signal's Case. The Series of the Even and Odd Parts

- x(t)- real signal; even $x_e(t)$ and odd part $x_o(t)$
- spectrum of real **even** signal $x_e(t)$ –real

$$x_e(t) = \frac{x(t)+x(-t)}{2} \xleftrightarrow{\mathcal{F}} \{\mathrm{Re}\, c_k^x\}$$

- spectrum of a real **odd** signal $x_o(t)$ - pure imaginary

$$x_o(t) = \frac{x(t)-x(-t)}{2} \xleftrightarrow{\mathcal{F}} \{j\,\mathrm{Im}\, c_k^x\}$$

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