Fourier Analysis of Discrete-Time Signals

1-Fourier series for discrete-time periodic signals
2-Discrete-Time Fourier Transform for aperiodic signals

- Discrete-time signals processing
- Fourier analysis
- Fast Fourier Transform algorithm

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The response of discrete-time linear invariant systems to the complex exponential with unitary magnitude

$e^{j\Omega_0 n} \longrightarrow$ H(Ω) $\longrightarrow e^{j\Omega_0 n} \cdot H(\Omega_0)$

Eigenfunction for any discrete time LTI

Eigenvalue

$$H(\Omega) = \sum_{k=-\infty}^{\infty} h[k] \cdot e^{-j\Omega k}$$

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- proof

$$y[n] = h[n] * e^{j\Omega_0 n} = \sum_{k=-\infty}^{\infty} h[k] \cdot e^{j\Omega_0 (n-k)} = e^{j\Omega_0 n} \cdot \sum_{k=-\infty}^{\infty} h[k] \cdot e^{-j\Omega_0 k}.$$

Notation: $H(\Omega) = \sum_{k=-\infty}^{\infty} h[k] \cdot e^{-j\Omega k}$

$$y[n] = e^{j\Omega_0 n} \cdot H(\Omega_0) = |H(\Omega_0)| \cdot e^{j[\Omega_0 n + \Phi(\Omega_0)]}$$

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linear combination of discrete-time complex exponentials with unitary magnitudes, different frequencies

⇒ response of the discrete-time LTI system - linear combination of partial responses, $H(\Omega_k)\ \Phi_k[n]$:

$$\sum_k a_k e^{j\Omega_k n} \longrightarrow \boxed{H(\Omega)} \longrightarrow \sum_k a_k H(\Omega_k) e^{j\Omega_k n}$$

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Fourier series for discrete-time periodic signals

$x[n]$ is <u>periodic of period N</u> if

$$x[n+N]=x[n], \text{ for any integer } n.$$

In a period N values $\{x[0], x[1], x[N-1]\}$.

$$x[0]= x[N],\ x[N+1]= x[1] \ldots$$

$$x[n]= x[(n)_N]$$

$(n)_N$ - n in the class of quotient (remainder) modulo N.
For $n<0$, the quotient $(n)_N$ positive.

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The space of periodic discrete-time signals of period N is N dimensional => N-dimensional basis. **Orthogonal basis**

$$\left\{ \Phi_k[n] = e^{jk\frac{2\pi}{N}n} \middle| k \in \aleph,\ 0 \le k \le N-1 \right\} \qquad \Omega_0 = \frac{2\pi}{N}$$

i.e. it's orthogonal and complete:

$$\left\langle \Phi_k[n], \Phi_l[n] \right\rangle = \begin{cases} N, & k = l \\ 0, & k \neq l \end{cases}$$

unique decomposition c_k.

$$x[n] = \sum_{k=0}^{N-1} c_k e^{jk\Omega_0 n},$$

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Proof

- Orthogonality

$$\langle \Phi_k[n], \Phi_l[n] \rangle = \sum_{n=0}^{N-1} e^{jk\Omega_0 n} \cdot e^{-jl\Omega_0 n} = \sum_{n=0}^{N-1} \left(e^{j(k-l)\Omega_0} \right)^n =$$

$$= \frac{1 - e^{j(k-l)\Omega_0 N}}{1 - e^{j(k-l)\Omega_0}} \overset{l \neq k}{=} \frac{1 - e^{j(k-l)2\pi}}{1 - e^{j(k-l)\Omega_0}}$$

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For *k=l:*

$$\langle \Phi_k[n], \Phi_k[n] \rangle = \sum_{n=0}^{N-1} e^{jk\Omega_0 n} \cdot e^{-jk\Omega_0 n} = \sum_{n=0}^{N-1} 1 = N.$$

For $k \neq l$:

$$\langle \Phi_k[n], \Phi_l[n] \rangle \overset{l \neq k}{=} \frac{1 - e^{j(k-l)2\pi}}{1 - e^{j(k-l)\Omega_0}} = 0.$$

$$\langle \Phi_k[n], \Phi_l[n] \rangle = \begin{cases} N, & k = l \\ 0, & k \neq l \end{cases} \text{ and } \left\| \Phi_k[n] \right\|_2^2 = N \text{ in } l^2[0, N-1].$$

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Completeness.

Any discrete-time periodic signal has the form:

$$x[n] = \sum_{k=0}^{N-1} c_k e^{jk\Omega_0 n},$$

with unique coefficients c_k. With the notations:

$$\phi_n = e^{j\Omega_0 n} \text{ and } \phi_n^k = e^{jk\Omega_0 n}$$

the last relation becomes for n=0, n=1,…,n=N-1

$$\begin{cases} x[0] = c_0 + c_1\phi_0 + c_2\phi_0^2 + ... + c_{N-1}\phi_0^{N-1} \\ x[1] = c_0 + c_1\phi_1 + c_2\phi_1^2 + ... + c_{N-1}\phi_1^{N-1} \\ x[2] = c_0 + c_1\phi_2 + c_2\phi_2^2 + ... + c_{N-1}\phi_2^{N-1} \\ \vdots \\ x[N-1] = c_0 + c_1\phi_{N-1} + c_2\phi_{N-1}^2 + ... + c_{N-1}\phi_{N-1}^{N-1} \end{cases}$$

-linear system of
-N equations and
-N unknowns c_0, c_1,…, c_{N-1}.

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The determinant of the eq.:

$$\Delta = \begin{vmatrix} 1 & \phi_0 & \phi_0^2 & \cdots & \phi_0^{N-1} \\ 1 & \phi_1 & \phi_1^2 & \cdots & \phi_1^{N-1} \\ 1 & \phi_2 & \phi_2^2 & \cdots & \phi_2^{N-1} \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \end{vmatrix} \overset{\text{Vandermonde}}{=}$$

$$(\phi_1 - \phi_0)(\phi_2 - \phi_0)\cdots(\phi_{N-1} - \phi_0)(\phi_2 - \phi_1)\cdots(\phi_{N-1} - \phi_{N-2}).$$

Representing the N discrete-time complex exponential with unitary magnitude in the complex plane, the following figure is obtained.

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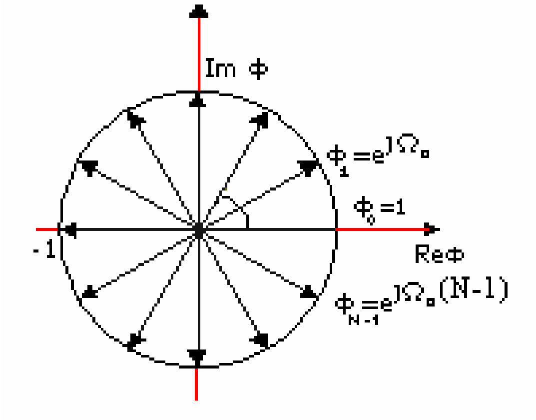

It can be seen that:

$\forall(k,l)$ for $k \neq l$ $\phi_k - \phi_l \neq 0$, so the determinant is not zero, $\Delta \neq 0$, and the system of equations has unique solution.

So, the considered set of complex exponentials is also complete. Being orthogonal and complete it is a basis.

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The decomposition of the periodic discrete-time signal $x[n]$ in the considered basis = decomposition in Fourier series:

$$c_k = \frac{\langle x[n], \phi_k[n] \rangle}{\| \phi_k[n] \|_2^2} = \frac{1}{N} \sum_{n=0}^{N-1} x[n] \cdot e^{-jk\Omega_0 n};$$

!!!definition

$$x[n] \leftrightarrow \{c_k\}$$

The **sequence of coefficients is also periodic** with same period.

$$c_{k+N} = c_k, \ 0 \leq k \leq N-1.$$

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- proof

$$c_{k+N} = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\cdot e^{-j(k+N)\frac{2\pi}{N}n} = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\cdot e^{-jk\frac{2\pi}{N}n}\cdot e^{-j2\pi n}$$

But $e^{-j2\pi n} = 1$ and $c_{k+N} = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\cdot e^{-jk\frac{2\pi}{N}n} = c_k,\ 0 \le k \le N-1.$

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Examples

1. The signal $x[n] = \sin\left(\frac{2\pi}{N}n\right)$

Euler's relation :

$$\sin\left(\frac{2\pi}{N}n\right) = \frac{1}{2j}e^{j\frac{2\pi}{N}n} - \frac{1}{2j}e^{-j\frac{2\pi}{N}n} = \frac{1}{2j}e^{j\frac{2\pi}{N}n} - \frac{1}{2j}e^{-j(N-1)\frac{2\pi}{N}n}.$$

Identification :

$$c_1 = \frac{1}{2j}\ ,\ c_{N-1} = -\frac{1}{2j}\ ;$$

$$c_0 = c_2 = ... = c_{N-2} = 0.$$

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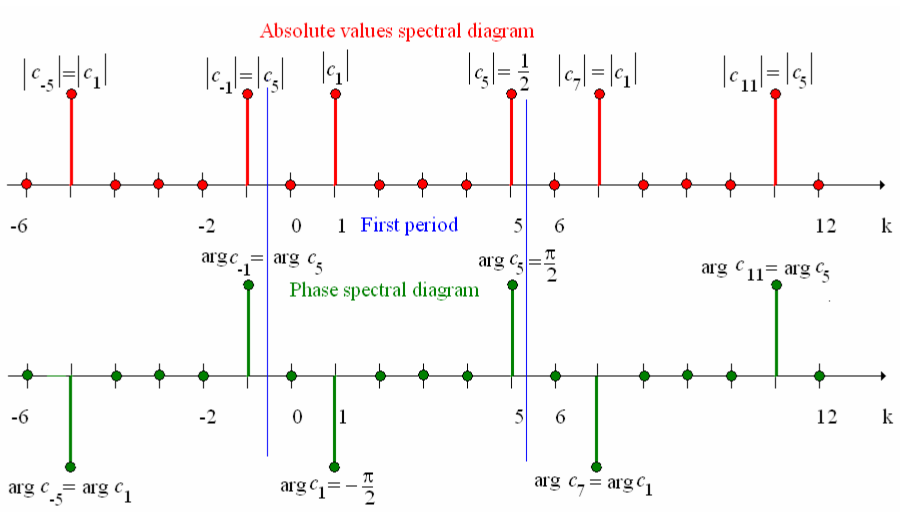
Spectral diagrams, N=6



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$$2.\ x[n]=1+\sin\left(\frac{2\pi}{N}n\right)+4\cos\left(\frac{2\pi}{N}n\right)+\cos\left(\frac{4\pi}{N}n+\frac{\pi}{2}\right)$$

Euler's relation :

$$x[n]=1+\frac{1}{2j}e^{j\frac{2\pi}{N}n}-\frac{1}{2j}e^{j(N-1)\frac{2\pi}{N}n}+2e^{j\frac{2\pi}{N}n}+2e^{j(N-1)\frac{2\pi}{N}n}+$$
$$+\frac{1}{2}e^{j\frac{\pi}{2}}e^{j2\frac{2\pi}{N}n}+\frac{1}{2}e^{-j\frac{\pi}{2}}e^{j(N-2)\frac{2\pi}{N}n}$$

identification:

$$c_0=1\ ;\ c_1=2+\frac{1}{2j}\ ;\ c_2=\frac{j}{2}\ ;\ c_{N-2}=-\frac{j}{2}\ ;\ c_{N-1}=2-\frac{1}{2j}.$$

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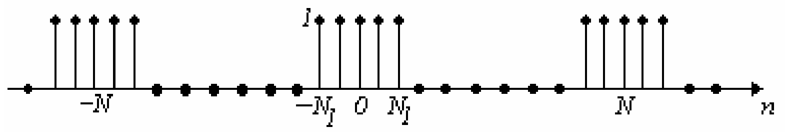
3. Periodic signal, period N.



definition:

$$c_k = \frac{1}{N}\sum_{n=-N_1}^{N_1} e^{-jk\frac{2\pi}{N}n} = \frac{1}{N} e^{jk\frac{2\pi}{N}N_1}\sum_{n=0}^{2N_1}\left(e^{-jk\frac{2\pi}{N}}\right)^n \quad ; 0 \le k \le N-1$$

For k=0:

$$c_0 = \frac{2N_1+1}{N} \ .$$

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$$c_k = \frac{1}{N} e^{jk\frac{2\pi}{N}N_1}\underbrace{\sum_{n=0}^{2N_1}\left(e^{-jk\frac{2\pi}{N}}\right)^n}_{\text{the sum of a geometric series}} \quad ; 0 \le k \le N-1$$

For $1 \le k \le N-1$:

$$c_k = \frac{1}{N}\cdot\frac{sin\left(k\frac{2\pi}{N}\frac{2N_1+1}{2}\right)}{sin\, k\frac{2\pi}{2N}} = \frac{1}{N}\cdot\left.\frac{sin\left[\left(2N_1+1\right)\frac{\Omega}{2}\right]}{sin\left(\frac{\Omega}{2}\right)}\right|_{\Omega=k\frac{2\pi}{N}} \quad ; 1 \le k \le N-1.$$

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•truncation of the Fourier series of a discrete-time signal, an approximation of this signal.

•better for higher number of terms.

•all *N* terms **no error**.

•example 3 , $N = 9$ and $2N_1+1 = 5$. Fourier coefficients :

$$c_0 = 0{,}556\ ,$$
$$c_1 = c_8 = 0{,}32\ ,$$
$$c_2 = c_7 = -0{,}059\ ,$$
$$c_3 = c_6 = -0{,}111, \text{ and}$$
$$c_4 = c_5 = 0{,}073.$$

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The truncated signal with $2M+1$ terms from its Fourier series:

$$x_M[n] = \sum_{k=-M}^{M} c_k e^{-jk\frac{2\pi}{9}n}$$

For M=1, 2, 3 and 4 :

$$x_1[n] = 0{,}556 + 0{,}64\cos\frac{2\pi}{9}n,$$
$$x_2[n] = 0{,}556 + 0{,}64\cos\frac{2\pi}{9}n - 0{,}118\cos\frac{4\pi}{9}n\ ,$$
$$x_3[n] = 0{,}556 + 0{,}64\cos\frac{2\pi}{9}n - 0{,}118\cos\frac{4\pi}{9}n - 0{,}222\cos\frac{6\pi}{9}n\ ,$$
$$x_4[n] = 0{,}556 + 0{,}64\cos\frac{2\pi}{9}n - 0{,}118\cos\frac{4\pi}{9}n - 0{,}222\cos\frac{6\pi}{9}n + 0{,}146\cos\frac{8\pi}{9}n.$$

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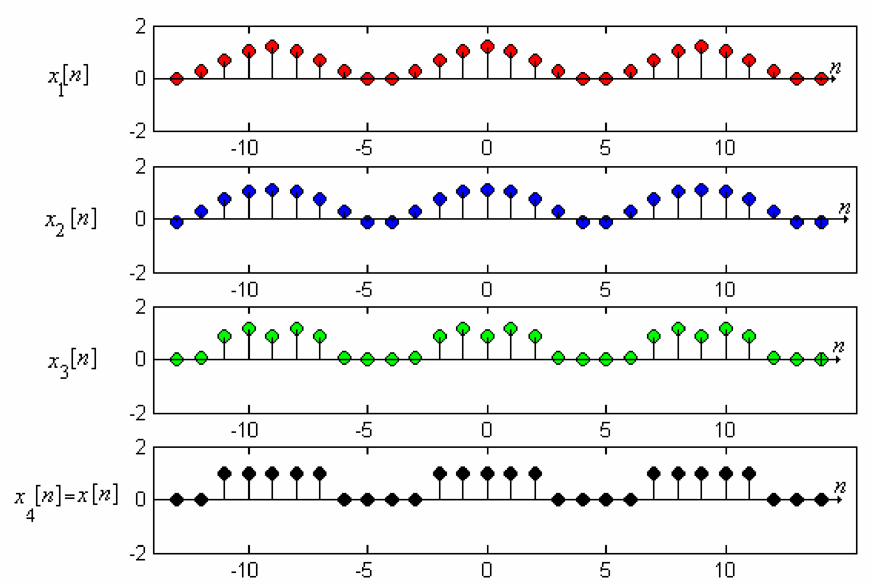

Properties of Discrete-Time Fourier Series

1. Linearity

$x[n]$ and $y[n]$ periodic with same period

Fourier coefficients c_k^x and c_k^y

$ax[n]+by[n]$ - same period.

$$x[n] \leftrightarrow \{c_k^x\} \; ; y[n] \leftrightarrow \{c_k^y\} \;\Rightarrow\; ax[n]+by[n] \leftrightarrow \{ac_k^x + bc_k^y\}$$

Homework: Prove it.

2. Time Shifting

$$x[n-n_0] \leftrightarrow \left\{ e^{-jk\frac{2\pi}{N}n_0} c_k^x \right\}$$

Dual operations: time shifting ↔ modulation in frequency (multiplication with a complex exponential).

The absolute value of the Fourier coefficients is not affected by time shifting.

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3. Complex conjugation

$$x[n] \in C \quad c_k^{x^*} = \left(c_{-k}^x\right)^*$$

Dual operations: complex conjugation in time ↔ reversal and complex conjugation in frequency domain.

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4. Time Reversal

$$x[-n] \leftrightarrow \{c^x_{-k}\}$$

Dual operations: time reversal ↔ reversal in frequency.

Reversal is auto-dual.

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5. Time scale modification

Different than in continuous-time domain (the scale factor can't be any real number).

One way :

$$x_{(m)}[n] = \begin{cases} x[n/m]\ ; & \text{if} \quad n \vdots m \\ 0\ ; & \text{in rest} \end{cases} \longleftrightarrow c_k^{x_{(m)}} = \frac{1}{m} c_k^x .$$

signal $x[n]$ periodic - period N, => signal $x_{(m)}[n]$ periodic $N' = mN$.

Proof.

$$x_{(m)}[n+N'] = \begin{cases} x[(n+N')/m]\ ; & \text{if} \quad n+N' \vdots m \\ 0\ ; & \text{in rest} \end{cases} \overset{N'=mN}{=} \begin{cases} x[n/m+N]\ ; & \text{if} \quad n \vdots m \\ 0\ ; & \text{in rest} \end{cases} =$$

$$= \begin{cases} x[n/m]\ ; & \text{if} \quad n \vdots m \\ 0\ ; & \text{in rest} \end{cases} = x_{(m)}[n].$$

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time dilation / interpolation : insert m-1 zeros between two consecutive samples of $x[n]$

$$x_{(m)}[n] = \begin{cases} x[n/m]\ ; & \text{if} \quad n \vdots m \\ 0\ ; & \text{in rest} \end{cases}$$

$\uparrow m$

The period of $x_{(m)}[n]$ is $N'=mN$, m times higher than the period of the signal $x[n]$

The fundamental frequency of the new signal - m times smaller.

Dual operations : discrete-time dilation ↔ **frequency compression** with the same scale factor.

The second way for the scale modification of a discrete-time signal is a **time compression** with information loss :

$$x^{(m)}[n] = x[mn]$$

$\downarrow m$

6. Signal's modulation

dual operation of **shifting**.

$$e^{jk_0\frac{2\pi}{N}n}x[n] \leftrightarrow \left\{c^x_{k-k_0}\right\}.$$

product of signal with the complex exponential = signal modulation.

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7. Signals' product

$$x[n]\cdot y[n] \leftrightarrow c^x_k \otimes c^y_k .$$

circular (or periodic) convolution of discrete-time periodic sequences, see next slide.

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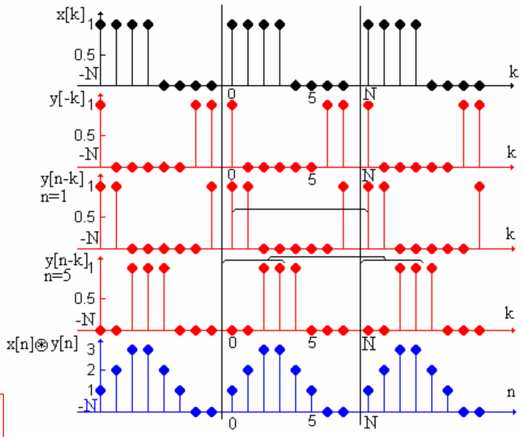
8. Circular convolution of signals

- dual operation of multiplication

$x[n]$ and $y[n]$ periodic with period N, Fourier coefficients c_k^x and c_k^y



$$z[n] = x[n] \otimes y[n] =$$

$$= \sum_{k=0}^{N-1} x[k]\, y\left[(n-k)_N\right].$$

$$x[n] \otimes y[n] \leftrightarrow \left\{ N c_k^x c_k^y \right\}.$$

The signal $z[n]$ is also periodic with period N.

$$z[n+N] = z[n].$$

9. Discrete-time Differentiation

defined by the first order difference of the signal $x[n]$ - difference between signal and its delayed version with 1.

If the original signal is periodic than its first order difference is also periodic with same period.

$$x[n] - x[n-1] \leftrightarrow \left\{ \left(1 - e^{-jk\frac{2\pi}{N}} \right) c_k^x \right\}.$$

Homework: Prove it.

10. Addition in time domain

The sum of samples of $x[n]$ with period N, with no DC component ($c_0^x = 0$) has the same period :

$$y[n] = \sum_{m=-\infty}^{n} x[m] \qquad y[n] \leftrightarrow \left\{ \frac{c_k^x}{1 - e^{-j\frac{2\pi}{N}k}} \right\}$$

Proof.

$$y[n+N] = \sum_{m=-\infty}^{n+N} x[m] = \sum_{m=-\infty}^{n} x[m] + \sum_{m=n+1}^{n+N} x[m]$$

sum of the samples over one period

$$c_k^x = \frac{1}{N}\sum_{n=0}^{N-1} x[n] \cdot e^{-jk\Omega_0 n};$$

For k=0:

$$c_0^x = \frac{1}{N}\sum_{n=0}^{N-1} x[n] = 0 \Rightarrow \sum_{m=n+1}^{n+N} x[m] = 0$$

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$$y[n] = \sum_{m=-\infty}^{n} x[m]$$

$$y[n-1] = \sum_{m=-\infty}^{n-1} x[m]$$

discrete-time differentiation of $y[n]$, $y[n]$-$y[n-1]$= $x[n]$:

$$y[n] \leftrightarrow \left\{ \frac{c_k^x}{1 - e^{-j\frac{2\pi}{N}k}} \right\}$$

For k=0 :

$c_0^y = 0.$

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11. Properties of the Fourier coefficients of the real signals

If $x[n]$ is real then it equals its complex conjugate. So its Fourier coefficients will equal the Fourier coefficients of its complex conjugate given by the property number 3.

$$x[n] \in R \Rightarrow x[n] = x^*[n] \Rightarrow c_k^x = \left(c_k^x\right)^* = \left(c_{-k}^x\right)^*$$

Polar form:

$$|c_k| = |c_{-k}| \; ; Arg\left\{c_k^x\right\} = -Arg\left\{c_{-k}^x\right\}$$

Cartesian form:

$$Re\left\{c_k^x\right\} = Re\left\{c_{-k}^x\right\} \; ; \; Im\left\{c_k^x\right\} = Im\left\{c_{-k}^x\right\}$$

The **even and odd** parts of a real signal:

$$x_p[n] \leftrightarrow \left\{Re\left\{c_k^x\right\}\right\} \; ; x_i[n] \leftrightarrow \left\{j\, Im\left\{c_k^x\right\}\right\}.$$

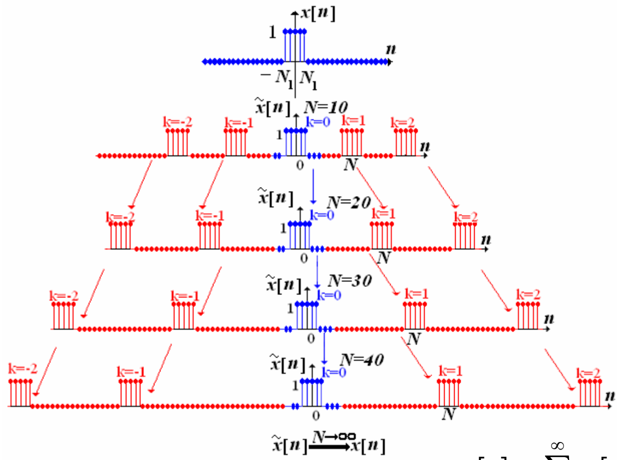
Parseval's Relation

$$\sum_{n=0}^{N-1} \left|x[n]\right|^2 = N \sum_{k=0}^{N-1} |c_k|^2 ;$$

the square of the l^2 norm of the considered signal.

-the power of the periodic signal - can be computed in the time domain or in the frequency domain.

Discrete-Time Fourier Transform



non-periodic signal → periodic signal by repetition: $\tilde{x}[n] = \sum_{k=-\infty}^{\infty} x[n-kN]$

$N \to \infty$, periodic signal → non-periodic one.

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Definition

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}$$ Discrete-Time Fourier Transform.

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(\Omega) e^{j\Omega n} d\Omega \;;$$ Inverse Discrete-Time Fourier Transform.

$$c_k = \frac{1}{N} X(k\Omega_0) \;,\; \Omega_0 = \frac{2\pi}{N} \;;$$

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The Fourier expansion of a periodic signal :

$$\tilde{x}[n] = \sum_{k\in<N>} c_k e^{jk\frac{2\pi}{N}n}$$

its Fourier coefficients are :

$$c_k = \frac{1}{N} \sum_{n\in<N>} \tilde{x}[n] e^{-jk\frac{2\pi}{N}n}$$

for the N values of k belonging to an interval which length is N the periodic signal and the original one are equals. For the rest of the values of n the original signal equals zero:

$$c_k = \frac{1}{N} \sum_{n=-\infty}^{\infty} x[n] e^{-jk\frac{2\pi}{N}n}$$

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$$c_k = \frac{1}{N} \sum_{n=-\infty}^{\infty} x[n] e^{-jk\frac{2\pi}{N}n} \qquad c_k = \frac{1}{N} X(k\Omega_0) \text{ , } \Omega_0 = \frac{2\pi}{N} ;$$

notation:

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n} ;$$

the corresponding periodic signal :

$$\tilde{x}[n] = \sum_{k\in<N>} \frac{1}{N} X(k\Omega_0) e^{jk\Omega_0 n} = \frac{1}{2\pi} \sum_{k\in<N>} X(k\Omega_0) e^{jk\Omega_0 n} \Omega_0$$

At the limit $N \to \infty$ ($\Omega_0 \to 0$) :

$$x[n] = \lim_{N\to\infty} \tilde{x}[n] = \frac{1}{2\pi} \int_{2\pi} X(\Omega) e^{j\Omega n} d\Omega ;$$

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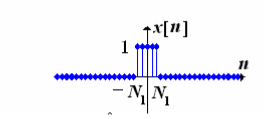
The periodic signal is:

$$\tilde{x}[n] = \frac{1}{2\pi} \sum_{k \in <N>} X(k\Omega_0) e^{jk\Omega_0 n} \Omega_0$$

where:

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n} ;$$

For the signal considered :



the function X(Ω) :

$$X(\Omega) = \frac{sin\left[(2N_1+1)\frac{\Omega}{2}\right]}{sin\frac{\Omega}{2}}.$$

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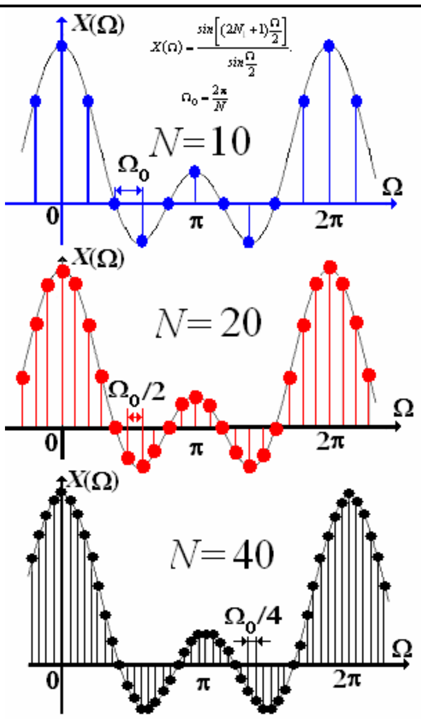

The spectrum of the original signal is the envelope $X(\Omega)$.

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$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}$$ Discrete-Time Fourier Transform.

If the signal $x[n]$ belongs to l^1 then the DTFT is convergent.

Proof.

$$|X(\Omega)| = \left| \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-jn\Omega} \right| \le \sum_{n=-\infty}^{\infty} |x[n]| \cdot \left| e^{-jn\Omega} \right| = \sum_{n=-\infty}^{\infty} |x[n]| = \|x[n]\|_1$$

The values of the DTFT of a signal from l^1 are finite.

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Initial Properties of the Discrete-Time Fourier Transform

1. **The Discrete-Time Fourier Transform of a signal is continuous.**

Proof.

For a little increment, $\Delta\Omega$, applying the definition of the Discrete-Time Fourier Transform, it can be written:

$$X(\Omega+\Delta\Omega) - X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n} \left(e^{-j\Delta\Omega n} - 1 \right)$$

The absolute value of this difference is bounded.

$$\left| \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-jn\Omega} \cdot \left(e^{-jn\Delta\Omega} - 1 \right) \right| \le \sum_{n=-\infty}^{\infty} |x[n]| \cdot \left(\left| e^{-jn\Delta\Omega} \right| + 1 \right)$$

$$0 \le |X(\Omega+\Delta\Omega) - X(\Omega)| \le 2 \|x[n]\|_1 \; ;$$

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$$0 \le \left| X(\Omega + \Delta\Omega) - X(\Omega) \right| \le 2 \left\| x[n] \right\|_1 \;;$$

Taking the limit for $\Delta\Omega \to 0$, we have:

$$0 \le \lim_{\Delta\Omega \to 0} \left| X(\Omega + \Delta\Omega) - X(\Omega) \right| \le$$

$$\le \sum_{n=-\infty}^{\infty} \left| x[n] \right| \lim_{\Delta\Omega \to 0} \left| e^{-j\Delta\Omega n} - 1 \right| = 0;$$

So:

$$\lim_{\Delta\Omega \to 0} X(\Omega + \Delta\Omega) = X(\Omega).$$

The Discrete-Time Fourier Transform of a signal is continuous.

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2. **The Discrete-Fourier Transform is periodic with period 2π.**

Applying the definition:

$$X(\Omega + 2\pi) = \sum_{n=-\infty}^{\infty} x[n] e^{-j(\Omega + 2\pi)n}$$

But:

$$e^{-j(\Omega+2\pi)n} = e^{-j\Omega n} \cdot e^{-j2\pi n} = e^{-j\Omega n}$$

So:

$$\boxed{X(\Omega + 2\pi) = X(\Omega);}$$

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The Case of Finite Energy Signals

$x[n] \in l^2$

In this case the Discrete-Time Fourier Transform is defined using the limit in mean square:

$$X(\Omega) = \underset{N\to\infty}{l.i.m} \sum_{n=-N}^{N} x[n] \cdot e^{-j\Omega n}.$$

limit in mean square *l.i.m.* :

$$\lim_{N\to\infty} \left\| X(\Omega) - \sum_{n=-N}^{N} x[n] \cdot e^{-j\Omega n} \right\|_2 = 0,$$

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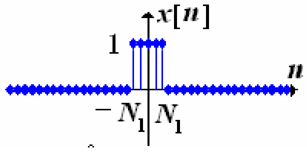
Examples

1. $x[n] = \sigma[n+N_1] - \sigma[n-N_1-1];$



$$X(\Omega) = \frac{\sin\left[(2N_1+1)\frac{\Omega}{2}\right]}{\sin\frac{\Omega}{2}}$$

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2.

$$x[n] = \delta[n] \;; \longrightarrow X(\Omega) = \sum_{n=-\infty}^{\infty} \delta[n] e^{-j\Omega n} = e^{0} = 1.$$

3. $x[n] = a^{n}\sigma[n]$, $|a| < 1,$

Applying the definition:

$$X(\Omega) = \sum_{n=-\infty}^{\infty} a^{n}\sigma[n] e^{-j\Omega n} = \sum_{n=0}^{\infty}\left(ae^{-j\Omega}\right)^{n} = \frac{1}{1 - ae^{-j\Omega}} \;, \; |a| < 1.$$

Complex function with the absolute value and the argument:

$$|X(\Omega)| = \frac{1}{\sqrt{1 - 2a\cos\Omega + a^{2}}} \;; Arg\left(X(\Omega)\right) = arctg\left(\frac{a\sin\Omega}{a\cos\Omega - 1}\right).$$

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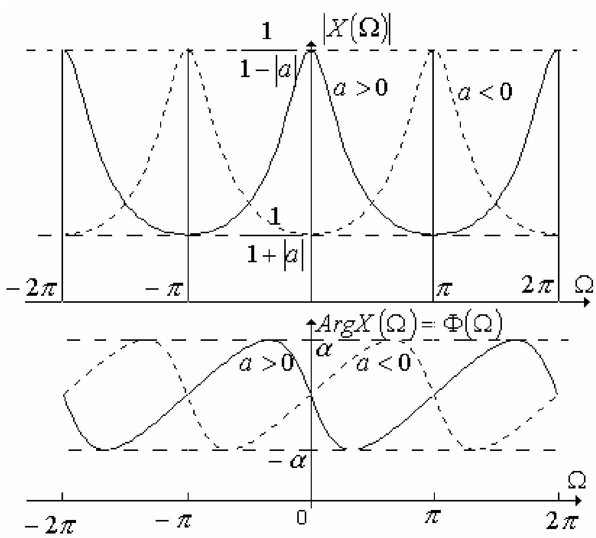
$$|X(\Omega)| = \frac{1}{\sqrt{1 - 2a\cos\Omega + a^{2}}} \;; Arg\left(X(\Omega)\right) = arctg\left(\frac{a\sin\Omega}{a\cos\Omega - 1}\right)$$



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Discrete Time Fourier transform for periodic signals

$$X(\Omega) = 2\pi \sum_{l=-\infty}^{\infty} c_l \delta\left(\Omega - l\frac{2\pi}{N}\right).$$

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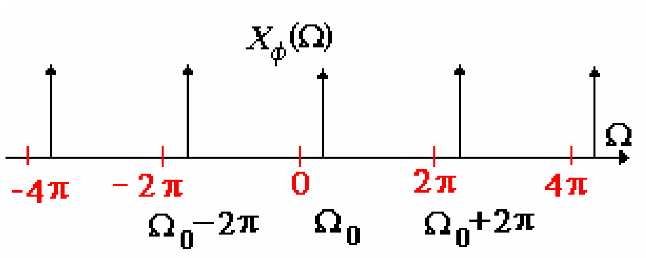
DTFT for periodic signals

The DTFT of the unitary magnitude complex exponential - expressed with the aid of distributions

$$\phi[n] = e^{j\Omega_0 n} \;; \longrightarrow X_\phi(\Omega) = 2\pi\delta_{2\pi}(\Omega - \Omega_0)$$



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Fourier series decomposition of a discrete-time signal :

$$x[n] = \sum_{k=0}^{N-1} c_k e^{jk\Omega_0 n} \ ;$$

for a complex exponential with the frequency $k\Omega_0$:

$$e^{jk\Omega_0 n} \leftrightarrow 2\pi\delta_{2\pi}(\Omega - k\Omega_0)$$

The DTFT is linear:

$$x[n] \leftrightarrow 2\pi \sum_{k=0}^{N-1} c_k \delta_{2\pi}(\Omega - k\Omega_0).$$

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$$x[n] \leftrightarrow 2\pi \sum_{k=0}^{N-1} c_k \delta_{2\pi}(\Omega - k\Omega_0).$$

Taking into account the definition of the Dirac's periodic distribution:

$$X(\Omega) = 2\pi \sum_{k=0}^{N-1} c_k \sum_{m=-\infty}^{\infty} \delta\left(\Omega - k\frac{2\pi}{N} - m \cdot 2\pi\right) = 2\pi \sum_{k=0}^{N-1} c_k \sum_{m=-\infty}^{\infty} \delta\left(\Omega - (k + mN)\frac{2\pi}{N}\right)$$

Let $l = k + mN$, on the basis of the Fourier coefficients periodicity it can be written, $c_l = c_k$. So the last relation becomes:

$$\boxed{X(\Omega) = 2\pi \sum_{l=-\infty}^{\infty} c_l \delta\left(\Omega - l\frac{2\pi}{N}\right).}$$

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An example

$$\delta_N[n] = \sum_{k=-\infty}^{\infty} \delta[n-kN]\,; \quad \xrightarrow{\text{The Fourier coefficients}} \quad c_k = \frac{1}{N}\sum_{n=0}^{N-1}\delta_N[n]e^{-jk\frac{2\pi}{N}n} = \frac{1}{N}$$

$$X(\Omega) = 2\pi \sum_{l=-\infty}^{\infty} c_l \delta\left(\Omega - l\frac{2\pi}{N}\right).$$

$$\Rightarrow \quad \boxed{\delta_N[n] \leftrightarrow 2\pi \sum_{k=-\infty}^{\infty} \frac{1}{N}\delta\left(\Omega - k\cdot\frac{2\pi}{N}\right) = \Omega_0 \cdot \delta_{\Omega_0}(\Omega).}$$

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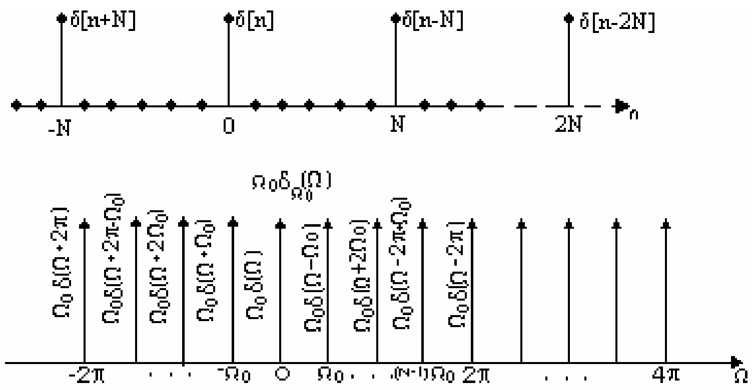
The original signal and the corresponding DTFT:



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Discrete-Time Fourier Transform

1. Linearity

$$ax[n]+by[n]\leftrightarrow aX(\Omega)+bY(\Omega).$$

2. Time-shifting

$$x[n-n_0]\leftrightarrow e^{-j\Omega n_0}X(\Omega).$$

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3. **Modulation in time**

$$e^{j\Omega_0 n}x[n]\leftrightarrow X(\Omega-\Omega_0).$$

4. **Time Scaling**

$$x_{(k)}[n]\leftrightarrow X(k\Omega).$$

5. **Signal's complex conjugation**

$$x^*[n]\leftrightarrow X^*(-\Omega).$$

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6. **Time reversal**

$$x[-n] \leftrightarrow X(-\Omega).$$

7. **Discrete-time differentiation**

$$x[n] - x[n-1] \leftrightarrow \left(1 - e^{-j\Omega}\right) X(\Omega).$$

8. **Signals convolution**

$$x[n] * y[n] \leftrightarrow X(\Omega) Y(\Omega).$$

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9. **Adding in time**

$$\sum_{k=-\infty}^{n} x[k] \leftrightarrow \frac{X(\Omega)}{1 - e^{-j\Omega}} + \pi X(0) \delta_{2\pi}(\Omega).$$

10. **Signals product**

$$x[n] y[n] \leftrightarrow \left(\frac{1}{2\pi}\right) \int_{2\pi} X(u) Y(\Omega - u)\, du = \left(\frac{1}{2\pi}\right) X(\Omega) \otimes Y(\Omega).$$

11. **Differentiation in frequency**

$$n x[n] \leftrightarrow j \frac{dX(\Omega)}{d\Omega}.$$

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12. Real Signal Discrete-Time Fourier Transforms' Properties

$x^*[n] = x[n] \Rightarrow X^*(-\Omega) = X(\Omega).$

$x_p[n] \leftrightarrow Re\{X(\Omega)\}$; $x_i[n] \leftrightarrow j\,Im\{X(\Omega)\}.$

$|X(\Omega)| = |X(-\Omega)|$; $Arg(X(\Omega)) = -Arg(X(-\Omega))$;

$Re\{X(\Omega)\} = Re\{X(-\Omega)\}$; $\mathrm{Im}\{X(\Omega)\} = -Im\{X(-\Omega)\}.$

13. Parseval's Relation

$x[n] \in l^2$, $\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} |X(\Omega)|^2 d\Omega$;

$\|X(\Omega)\|^2_{L^2_{[-\pi,\pi]}} = 2\pi \|x[n]\|^2_{l^2}$.

$\forall x[n], y[n] \in l^2$, $\langle X(\Omega), Y(\Omega)\rangle_{L^2_{[-\pi,\pi]}} = 2\pi \langle x[n], y[n]\rangle_{l^2}$

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Energy Spectral Density

$S_X(\Omega)$ - square of the absolute value of the Discrete-Time Fourier Transform. The energy of the considered signal is obtained integrating it.

$S_X(\Omega) = |X(\Omega)|^2$; =Energy spectral density

$W = \frac{1}{2\pi}\int_{2\pi} S_X(\Omega)\, d\Omega.$ =Energy

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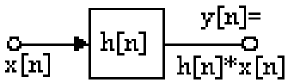
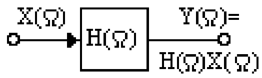
Frequency Response of Discrete-Time Linear Invariant Systems





$$h[n] \leftrightarrow H(\Omega).$$

The Discrete-Time Fourier Transform of the impulse response of an LTI system gives the frequency response of that system.

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The Response of a Discrete-Time LTI system to a Periodic Discrete-Time Input Signal

$$x[n] = \sum_{k=0}^{N-1} c_k e^{jk\Omega_0 n} \Rightarrow y[n] = \sum_{k=0}^{N-1} c_k H(k\Omega_0) e^{jk\Omega_0 n};$$

Particular example, harmonic input signal:

$$x[n] = A\cos\left(\frac{2\pi}{N} + \varphi_0\right) \longrightarrow y[n] = A\left|H(\Omega_0)\right|\cos\left[\Omega_0 n + \Phi(\Omega_0) + \varphi_0\right].$$

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$$x[n]=\frac{A}{2}\left(e^{j\left(\frac{2\pi}{N}+\varphi_0\right)}+e^{-j\left(\frac{2\pi}{N}+\varphi_0\right)}\right);$$

Fourier coefficients: $c_1=\frac{A}{2}e^{j\varphi_0}$ $c_{N-1}=\frac{A}{2}e^{-j\varphi_0}$.

$$y[n]=\frac{A}{2}e^{j\varphi_0}H(\Omega_0)e^{j\Omega_0 n}+\frac{A}{2}e^{-j\varphi_0}H(-\Omega_0)e^{-j\Omega_0 n};$$

$H(\Omega)=H^*(-\Omega)$. because $h[n]\in R$

$$y[n]=\frac{A}{2}\left|H(\Omega_0)\right|e^{j\left[\Omega_0 n+\Phi(\Omega_0)+\varphi_0\right]}+\frac{A}{2}\left|H(\Omega_0)\right|e^{-j\left[\Omega_0 n+\Phi(\Omega_0)+\varphi_0\right]},$$

$$y[n]=A\left|H(\Omega_0)\right|cos\left[\Omega_0 n+\Phi(\Omega_0)+\varphi_0\right].$$

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The Case of LTI Systems Described by Linear Finite Difference Equations with Constant Coefficients

The general form of a linear finite difference equation with constant coefficients is:

$$\sum_{k=0}^{N}a_k y[n-k]=\sum_{k=0}^{N}b_k x[n-k],\ a_0\neq 0,$$

Taking the Discrete-Time Fourier Transform of the both sides it results:

$$Y(\Omega)\sum_{k=0}^{N}a_k\left(e^{-j\Omega}\right)^k=X(\Omega)\sum_{k=0}^{N}b_k\left(e^{-j\Omega}\right)^k,\ a_0\neq 0.$$

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$$Y(\Omega)\sum_{k=0}^{N} a_k\left(e^{-j\Omega}\right)^k = X(\Omega)\sum_{k=0}^{N} b_k\left(e^{-j\Omega}\right)^k,\ a_0 \neq 0.$$

The frequency response of the system is obtained dividing the Discrete-Time Fourier transform of the output by the Discrete-Time Fourier Transform of the input:

$$H(\Omega) = \frac{Y(\Omega)}{X(\Omega)} = \frac{\sum_{k=0}^{M} b_k\left(e^{-j\Omega}\right)^k}{\sum_{k=0}^{N} a_k\left(e^{-j\Omega}\right)^k}\ ;\ a_0 \neq 0.$$

Examples

i) $$y[n] - \frac{\sqrt{2}}{2} y[n-1] + \frac{1}{4} y[n-2] = x[n] - x[n-1]\ .$$

Identifying this equation with the general form:

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{N} b_k x[n-k],\ a_0 \neq 0\ ,$$

the coefficients a_k and b_k are obtained. By their substitution in the general expression of the frequency response:

$$H(\Omega) = \frac{Y(\Omega)}{X(\Omega)} = \frac{\sum_{k=0}^{M} b_k\left(e^{-j\Omega}\right)^k}{\sum_{k=0}^{N} a_k\left(e^{-j\Omega}\right)^k}\ ;\ a_0 \neq 0.$$

$$H(\Omega)=\frac{1-e^{-j\Omega}}{1-\frac{\sqrt{2}}{2}e^{-j\Omega}+\frac{1}{4}e^{-2j\Omega}}$$

$$H(\Omega)=\frac{1-e^{-j\Omega}}{\left(1-\alpha_1 e^{-j\Omega}\right)\left(1-\alpha_2 e^{-j\Omega}\right)}, \text{ with: } \alpha_{1,2}=\frac{\sqrt{2}}{4}(1\pm j)=\frac{1}{2}e^{\pm j\frac{\pi}{4}}.$$

To determine the impulse response, the frequency response must be decomposed in simple fractions:

$$H(\Omega)=\frac{A_1}{1-\alpha_1 e^{-j\Omega}}+\frac{A_1}{1-\alpha_2 e^{-j\Omega}}\ ; A_{1,2}=\frac{1}{2}\pm j\frac{2\sqrt{2}-1}{2}.$$

Applying for each simple fraction the Inverse Discrete-Time Fourier Transform, the impulse response can be obtained:

$$h[n]=\left(A_1\alpha_1^n+A_2\alpha_2^n\right)\sigma[n]=...=\frac{\sqrt{2}}{2^n}\left(cos\frac{n+1}{4}\pi-2sin\frac{n\pi}{4}\right)\sigma[n].$$

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ii) determine the impulse response of the system from the frequency response:

$$H(\Omega)=\frac{3}{\left(1-\frac{1}{2}e^{-j\Omega}\right)\left(1-\frac{1}{8}e^{-j\Omega}\right)}.$$

decomposition in simple fractions:

$$H(\Omega)=\frac{4}{1-\frac{1}{2}e^{-j\Omega}}-\frac{1}{1-\frac{1}{8}e^{-j\Omega}}\ ;$$

the Inverse Discrete-Time Fourier Transform :

$$h[n]=\left(4\frac{1}{2^n}-\frac{1}{2^{3n}}\right)\sigma[n].$$

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iii) the impulse response and the unit step response for :

$$y[n]-ay[n-1]=x[n]\ ,\ |a|<1.$$

Identifying the coefficients and applying the general relation :

$$H(\Omega)=\frac{1}{1-ae^{-j\Omega}}\ \Rightarrow\ h[n]=a^n\sigma[n]\ .$$

The Discrete-Time Fourier Transform of the unit step signal is:

$$X_\sigma(\Omega)=\frac{1}{1-e^{-j\Omega}}+\pi\delta_{2\pi}(\Omega),$$

The Discrete-Time Fourier Transform of the response becomes:

$$S(\Omega)=H(\Omega)X_\sigma(\Omega)\ ;$$

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The Discrete-Time Fourier Transform of the step response :

$$S(\Omega)=\frac{1}{\left(1-ae^{-j\Omega}\right)\left(1-e^{-j\Omega}\right)}+\frac{\pi}{1-a}\delta_{2\pi}(\Omega).$$

decomposition in simple fractions :

$$S(\Omega)=...=-\frac{a}{1-a}\frac{1}{1-ae^{-j\Omega}}+\frac{1}{1-a}\left[\frac{1}{1-e^{-j\Omega}}+\pi\delta_{2\pi}(\Omega)\right];$$

Inverse Discrete-Time Fourier Transform ⇒ the unit step response of the considered system:

$$s[n]=-\frac{a}{1-a}a^n\sigma[n]+\frac{1}{1-a}\sigma[n]=\frac{1-a^{n+1}}{1-a}\sigma[n].$$

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First and Second Order Discrete LTI Systems

the frequency response of a discrete LTI system is a rational function.

$$H(\Omega)=\frac{\sum_{k=0}^{M} b_k\left(e^{-j\Omega}\right)^k}{\sum_{k=0}^{N} a_k\left(e^{-j\Omega}\right)^k}$$

The two polynomials can be factorized. Each factor is a first or a second order polynomial.

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$$H(\Omega)=\frac{b_0}{a_0}\frac{\prod_{k=1}^{P}\left(1+\beta_{1k}e^{-j\Omega}+\beta_{2k}e^{-2j\Omega}\right)\prod_{k=1}^{M-2P}\left(1+\mu_k e^{-j\Omega}\right)}{\prod_{k=1}^{Q}\left(1+\alpha_{1k}e^{-j\Omega}+\alpha_{2k}e^{-2j\Omega}\right)\prod_{k=1}^{N-2Q}\left(1+\eta_k e^{-j\Omega}\right)}$$

-product in frequency domain = a convolution in time domain $\Rightarrow$

first order polynomial factor $\Leftrightarrow$ first order system

second order polynomial factor $\Leftrightarrow$ second order system.

-knowing the behaviour in frequency of first and second order systems we can deduce the behaviour in frequency of any discrete LTI system.

This is why the first and second order systems are so important.

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$$H(\Omega)=\frac{b_0}{a_0}\frac{\prod_{k=1}^{P}\left(1+\beta_{1k}e^{-j\Omega}+\beta_{2k}e^{-2j\Omega}\right)\prod_{k=1}^{M-2P}\left(1+\mu_k e^{-j\Omega}\right)}{\prod_{k=1}^{Q}\left(1+\alpha_{1k}e^{-j\Omega}+\alpha_{2k}e^{-2j\Omega}\right)\prod_{k=1}^{N-2Q}\left(1+\eta_k e^{-j\Omega}\right)}$$

For $M=N$, the decomposition of the frequency response in simple fractions is:

$$H(\Omega)=\frac{b_N}{a_N}+\sum_{k=1}^{Q}\frac{\gamma_{0k}+\gamma_{1k}e^{-j\Omega}}{1+\alpha_{1k}e^{-j\Omega}+\alpha_{2k}e^{-2j\Omega}}+\sum_{k=1}^{N-2Q}\frac{A_k}{1+\eta_k e^{-j\Omega}}$$

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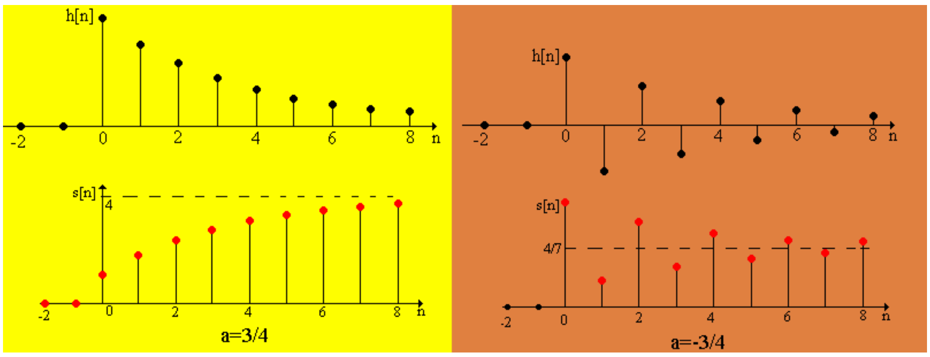
First Order Systems

$$y[n]-ay[n-1]=x[n]\ ,\ |a|<1\ .\quad H(\Omega)=\frac{1}{1-ae^{-j\Omega}}\ ;$$

$$h[n]=a^n\sigma[n]\ .\qquad s[n]=\frac{1-a^{n+1}}{1-a}\sigma[n].$$



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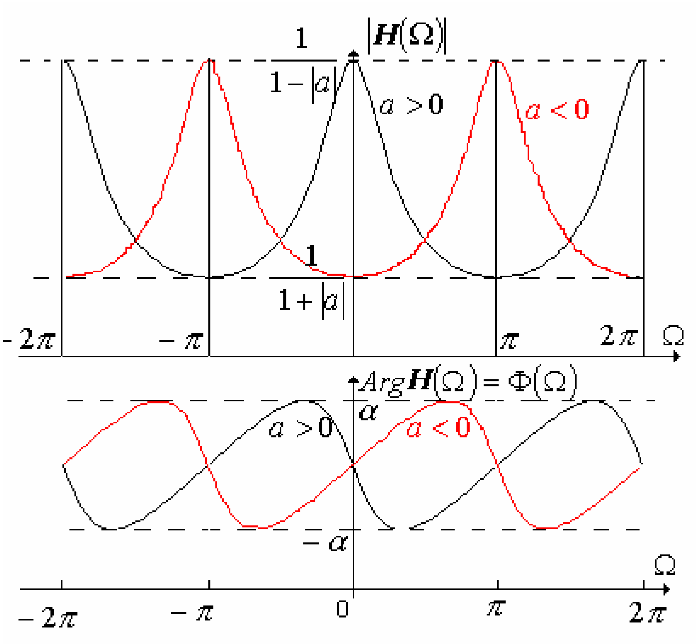

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Second Order Systems

the linear finite difference equation - constant coefficients and two parameters:

$$y[n]-2\rho\cos\theta\, y[n-1]+\rho^2 y[n-2]=x[n]\ ,\ 0<\rho<1\ ,\ 0\le\theta\le\pi,$$

The frequency response:

$$H(\Omega)=\frac{1}{1-2\rho\cos\theta e^{-j\Omega}+\rho^2 e^{-2j\Omega}}=\frac{1}{\left(1-\rho e^{j\theta}e^{-j\Omega}\right)\left(1-\rho e^{-j\theta}e^{-j\Omega}\right)}$$

The impulse response:

$$h[n]=\frac{e^{j\theta}}{2j\sin\theta}\left[e^{j\theta}\rho^n e^{jn\theta}-e^{-j\theta}\rho^n e^{-jn\theta}\right]\sigma[n]=\rho^n\frac{\sin(n+1)\theta}{\sin\theta}\sigma[n]\ ,\ \theta\in(0,\pi).$$

The unit step response:

$$s[n]=\left[\frac{e^{j\theta}}{2j\sin\theta}\frac{1-\left(\rho e^{j\theta}\right)^{n+1}}{1-\rho e^{j\theta}}-\frac{e^{-j\theta}}{2j\sin\theta}\frac{1-\left(\rho e^{-j\theta}\right)^{n+1}}{1-\rho e^{-j\theta}}\right]\sigma[n]\ ;\ \theta\in(0,\pi).$$

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Two Particular Cases

1.

$\theta = 0.$

$$H(\Omega) = \frac{1}{\left(1-\rho e^{-j\theta}\right)^2} \;\Rightarrow\; h[n] = (n+1)\rho^n \sigma[n] \;.$$

$$S(\Omega) = -\frac{\rho}{1-\rho}\frac{1}{\left(1-\rho e^{-j\Omega}\right)^2} - \frac{\rho}{(1-\rho)^2}\frac{1}{1-\rho e^{-j\Omega}} + \frac{1}{(1-\rho)^2}\frac{1}{1-e^{-j\Omega}} + \frac{\pi}{(1-\rho)^2}\delta_{2\pi}(\Omega).$$

$$s[n] = \left[\frac{1}{(1-\rho)^2} - \frac{\rho}{(1-\rho)^2}\rho^n - \frac{\rho}{1-\rho}(n+1)\rho^n\right]\sigma[n].$$

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2.

$\theta = \pi.$

$$H(\Omega) = \frac{1}{\left(1+\rho e^{-j\Omega}\right)^2}.$$

$$h[n] = (n+1)(-\rho)^n \sigma[n] \;,$$

$$s[n] = \left[\frac{1}{(1+\rho)^2} + \frac{\rho}{(1+\rho)^2}(-\rho)^n + \frac{\rho}{1+\rho}(n+1)(-\rho)^n\right]\sigma[n].$$

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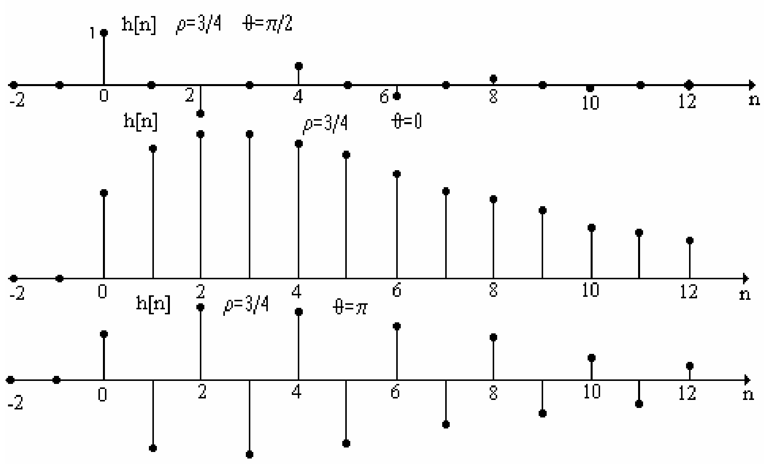
Some Graphical Representations



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Frequency Characteristics

$$\left|H(\Omega)\right| = \frac{1}{\sqrt{4\rho^2 \cos^2\Omega - 4\rho\left(1+\rho^2\right)\cos\theta\cos\Omega + \left(1-\rho^2\right)^2 + 4\rho^2\cos^2\theta}}.$$

$$\Phi(\Omega) = -arctg\,\frac{2r\sin\theta\left(a - r\cos\Omega\right)}{1 - 2r\cos\theta\cos\Omega + r^2\cos 2\Omega}.$$

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