1. Continuous- and discrete- time signals

1. Complex numbers

For any complex number, $z \in \mathbb{C}$, we have

- **Cartesian/algebraic form**

$$z = x + jy; \quad j = \sqrt{-1}$$

where

$$x = \mathrm{Re}\{z\}$$
$$y = \mathrm{Im}\{z\}$$

- **Polar form**

$$z = r \cdot e^{j\theta}; \quad j = \sqrt{-1}$$

Modulus argument/phase

$$r = |z| \qquad \arg\{z\} = \measuredangle\{z\}$$

The **complex conjugate** of a complex number:

$$z^* = x - jy = r \cdot e^{-j\theta}$$

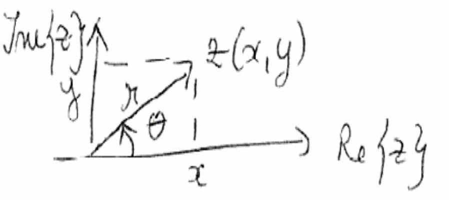
The **absolute value**

$$|z| = \sqrt{x^2 + y^2}$$

$$z \cdot z^* = |z|^2$$

Relation between polar and cartesian coordinates

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$



$$\begin{cases} r = \sqrt{x^2 + y^2} \\ \theta = \begin{cases} \mathrm{arctg}\dfrac{y}{x} & x > 0 \\ \mathrm{arctg}\dfrac{y}{x} + \pi & x < 0, y \geq 0 \\ \mathrm{arctg}\dfrac{y}{x} - \pi & x < 0, y < 0 \end{cases} \end{cases} \qquad \theta = \begin{cases} \dfrac{\pi}{2} & x = 0, y > 0 \\ -\dfrac{\pi}{2} & x = 0, y < 0 \end{cases}$$

Euler's relation

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$\begin{cases} \cos\theta = \dfrac{e^{j\theta} + e^{-j\theta}}{2} \\ \sin\theta = \dfrac{e^{j\theta} - e^{-j\theta}}{2j} \end{cases}$$

Problems

P1. Determine the Cartesian coordinates of the following complex numbers, depending on x_0 and y_0, where we have:

$$z_0 = x_0 + jy_0$$

a) $z_1 = r_0 \cdot e^{-j\theta_0}$

b) $z_2 = r_0$

c) $z_3 = r_0 \cdot e^{j(\theta_0+\pi)}$

d) $z_4 = r_0 \cdot e^{-j(\theta_0+\pi)}$

e) $z_5 = r_0 \cdot e^{j(\theta_0+2\pi)}$

Sketch $z_1, z_2, ..., z_5$ in the complex plane for the following cases of r_0 and θ_0

$$r_0 = 2 \quad \theta_0 = \frac{\pi}{4}$$

$$r_0 = 2 \quad \theta_0 = \frac{\pi}{2}$$

Solution

a)

$$z_1 = r_0 \cdot e^{-j\theta_0} = r_0 \cdot \left[\cos(-\theta_0) + j\sin(-\theta_0)\right]$$
$$= r_0 \cdot \left[\cos\theta_0 - j\sin\theta_0\right]$$
$$= x_0 - jy_0$$

b)

$$z_2 = r_0 = \sqrt{x_0^2 + y_0^2}$$

c)

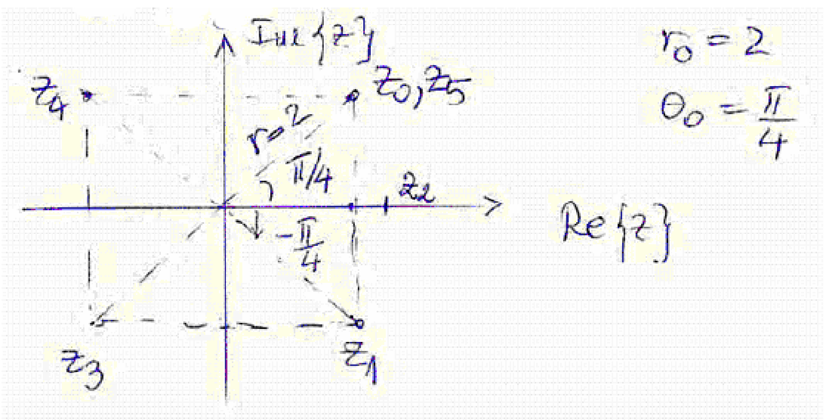
$$z_3 = r_0 \cdot e^{j(\theta_0+\pi)} = r_0 \cdot \left[\cos(\theta_0+\pi) + j\sin(\theta_0+\pi)\right]$$
$$= r_0 \cdot \left[-\cos\theta_0 - j\sin\theta_0\right] = -x_0 - jy_0$$

d)

$$z_4 = r_0 \cdot e^{-j(\theta_0+\pi)} = -x_0 + jy_0$$

e)

$$z_5 = r_0 \cdot e^{j(\theta_0+2\pi)} = x_0 + jy_0$$



P2. Consider z a complex variable with its complex conjugate

$$z^* = x - jy = r \cdot e^{-j\theta}$$

Prove the following relations are true, where z, z_1 and z_2 are complex numbers.

a) $z \cdot z^* = r^2$

b) $\dfrac{z}{z^*} = e^{2j\theta}$

c) $z + z^* = 2\,\mathrm{Re}\{z\}$

d) $z - z^* = 2j\,\mathrm{Im}\{z\}$

e) $(z_1 + z_2)^* = z_1^* + z_2^*$

f) $(az_1 z_2)^* = az_1^* z_2^*$

g) $\left(\dfrac{z_1}{z_2}\right)^* = \dfrac{z_1^*}{z_2^*}$

Solution

a) $z \cdot z^* = r \cdot e^{j\theta} \cdot r \cdot e^{-j\theta} = r^2$

b) $\dfrac{z}{z^*} = \dfrac{r \cdot e^{j\theta}}{r \cdot e^{-j\theta}} = e^{2j\theta}$

c) $z + z^* = x + jy + x - jy = 2x = 2\,\mathrm{Re}\{z\}$

d) $z - z^* = x + jy - x + jy = 2jy = 2j\,\mathrm{Im}\{z\}$

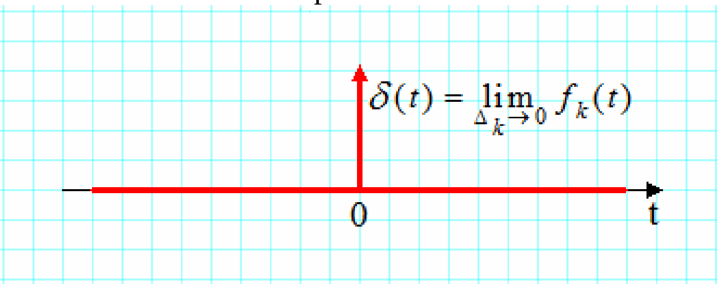
e) $(z_1 + z_2)^* = (x_1 + jy_1 + x_2 + jy_2)^* = x_1 + x_2 - j(y_1 + y_2) = z_1^* + z_2^*$

f) $(az_1 z_2)^* = \left(a \cdot r_1 e^{j\theta_1} \cdot r_2 e^{j\theta_2}\right)^* = ar_1 r_2 e^{-j(\theta_1+\theta_2)} = a \cdot r_1 e^{-j\theta_1} r_2 e^{-j\theta_2} = az_1^* z_2^*$

g) $\left(\dfrac{z_1}{z_2}\right)^* = \dfrac{z_1^*}{z_2^*}$

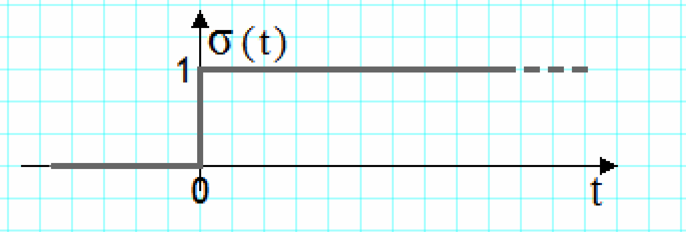
Continuous-time signals

Continuous-time unit impulse



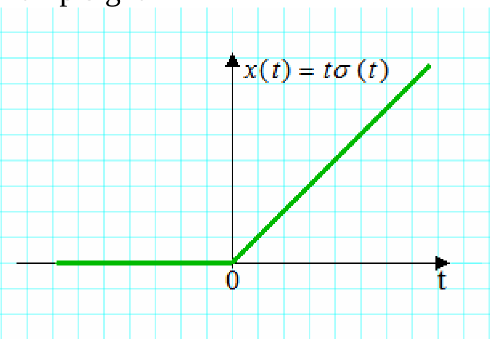
$$\delta(t)=\begin{cases}\infty, & t=0\\ 0, & t\neq 0\end{cases}$$

Continuous-time unit step

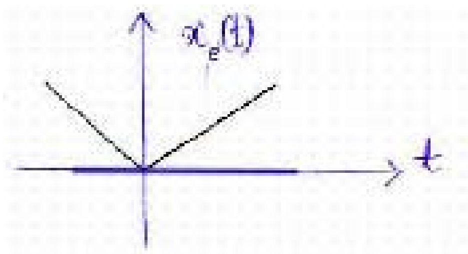


$$\sigma(t)=\begin{cases}1, & t\geq 0\\ 0, & t<0\end{cases}$$

Ramp signal

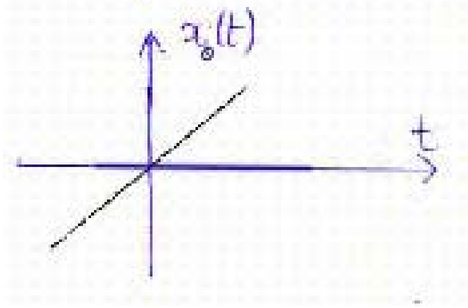


$$r(t)=\begin{cases}t, & t\geq 0\\ 0, & t<0\end{cases}=t\sigma(t)$$



Even signal $x_e(t)=\dfrac{x(t)+x(-t)}{2}$

$x_e(-t)=x_e(t)$



Odd signal $x_o(t)=\dfrac{x(t)-x(-t)}{2}$

$x_o(-t)=-x_o(t)$

Any given signal x(t) has an odd part and an even part

$$x(t) = x_e(t) + x_o(t)$$

With

$$x_e(t) = \frac{x(t)+x(-t)}{2} \text{ and } x_o(t) = \frac{x(t)-x(-t)}{2}$$

- ***Energy*** of a continuous-time signal

$$W = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

If the signals has the support $[t_1, t_2]$, the energy is:

$$W = \int_{t_1}^{t_2} |x(t)|^2 dt$$

Power of the continuous-time signal (again support $[t_1, t_2]$)

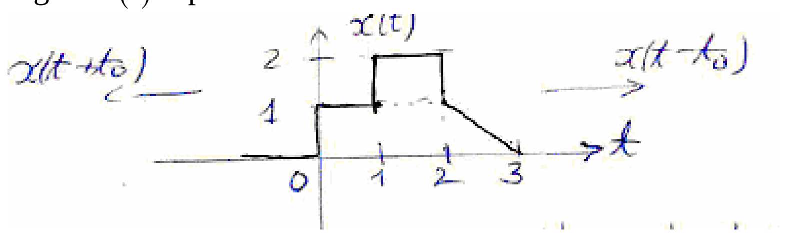
$$P = \frac{W}{t_2 - t_1} = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} |x(t)|^2 dt$$

For periodic signals the power is computed over one period!!!

$$P = \frac{1}{T_0} \int_{T_0} |x(t)|^2 dt$$

Problems

P3. Consider the signal $x(t)$ represented below.

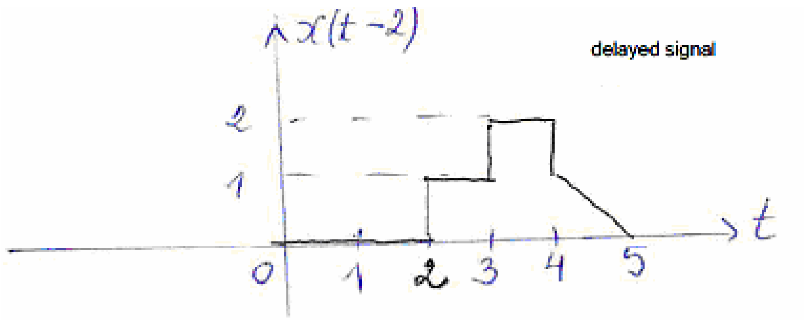


Sketch the following signals:

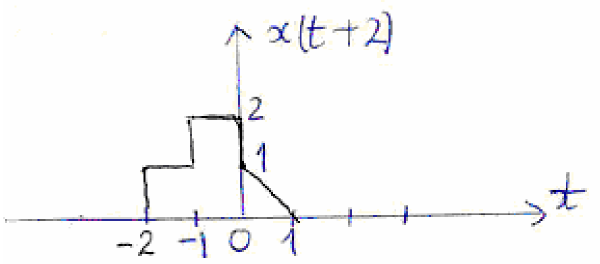
a) $x(t-2)$	e) $x(-t-1)$	i) $x(t)\cdot\delta(t)$
b) $x(t+2)$	f) $x(2t)$	j) $x(t)\cdot[\sigma(t-1)-\sigma(t-2)]$
c) $x(-t)$	g) $x(t/2)$	
d) $x(-t+3)$	h) $x(2t-4)$	

Solution.

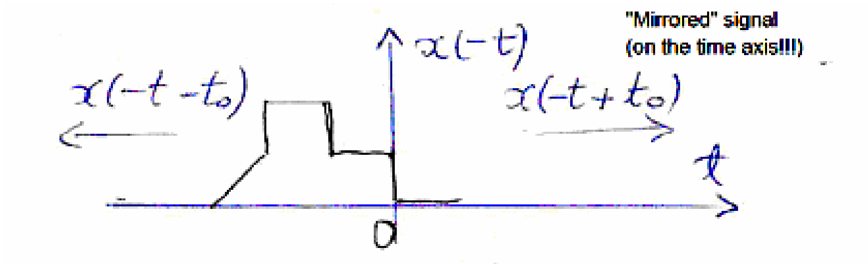
a) $x(t-2)$. The "new origin" is $t=2$: $t-2=0$. This is a delayed version of the signal



b) $x(t+2)$. The “new origin” is $t=-2$: $x(t+2) \Rightarrow t+2=0$

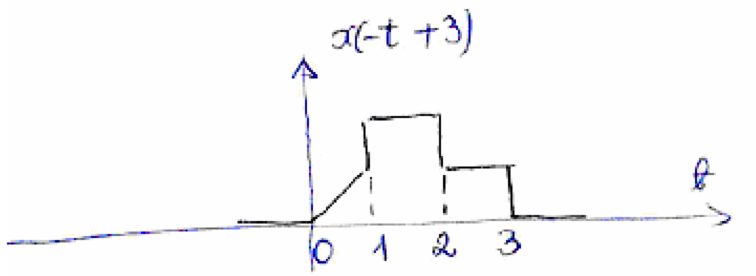


c) $x(-t)$. The signal is reversed or “mirrored” on the time axis

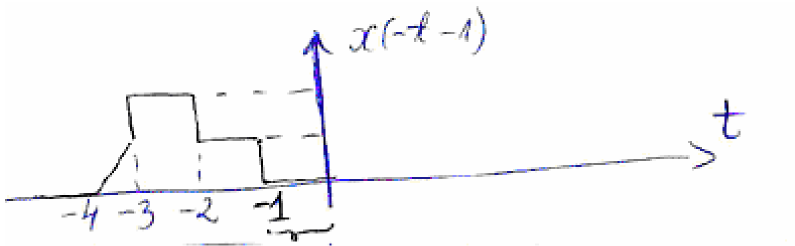


d) $x(-t+3)$

same as c) with shifting on the time axis. The origin is shifted on $t=3$

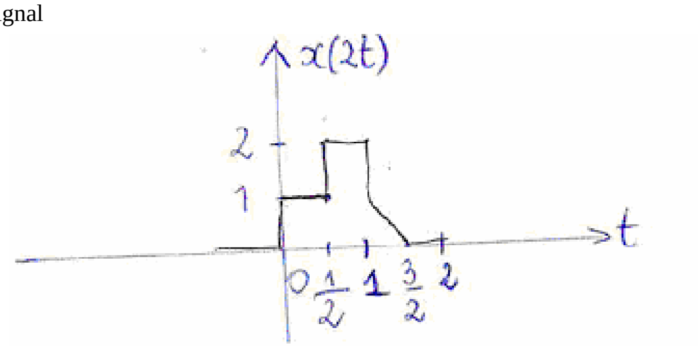


e) $x(\underbrace{-t-1}_{-(t+1)})$



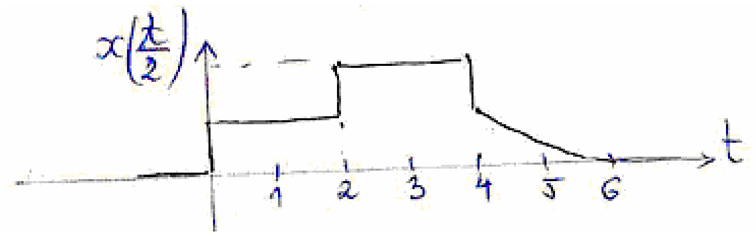
f) $x(2t)$. The support is compressed; this is signal compression in time.

$2\underbrace{t}_{\text{new signal}} = \underbrace{t'}_{\text{original signal}}$; for $2t = t' = 1$; on the new time axis we have $t = 1/2$ and so on.

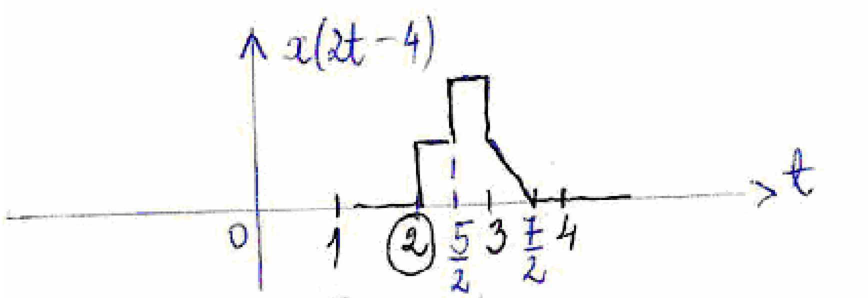


g) $x(t/2)$. The support is dilated; this is signal dilation in time.

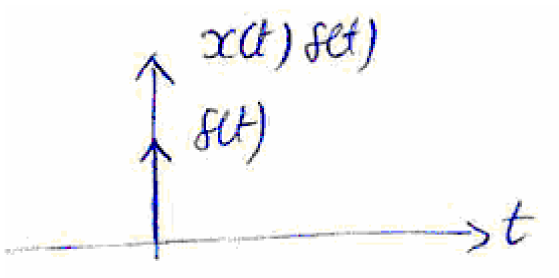
$\underbrace{t}_{\text{new signal}}/2=\underbrace{t'}_{\text{original signal}}$; for $t/2=t'=1$; on the new time axis we have $t=2$ and so on.



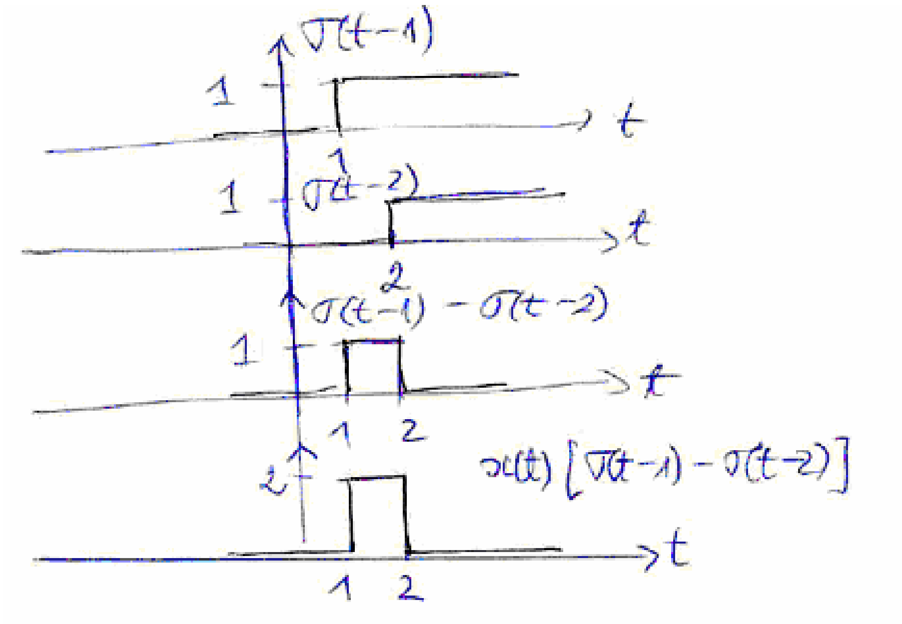
h) $x(2t-4)$. $2t-4=0 \Rightarrow t=2$. The origin is now in $t=2$ (time compression + shifting)



i) $x(t)\cdot\delta(t)$

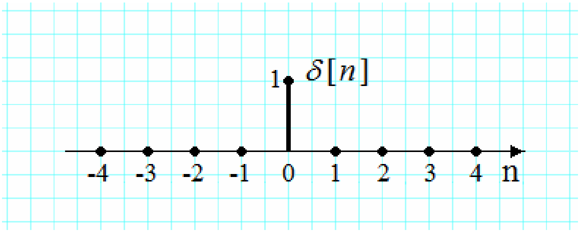


j) $x(t)\cdot[\sigma(t-1)-\sigma(t-2)]$



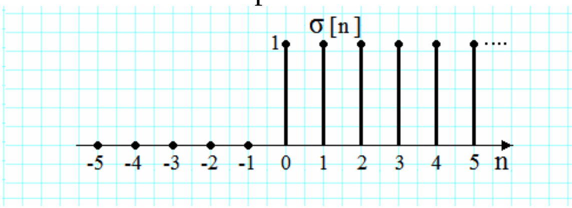
Discrete-time signals

Discrete-time unit impulse (Dirac impulse)



$$\delta[n]=\begin{cases}1, & n=0\\ 0, & n\neq 0\end{cases}$$

Discrete-time unit step



$$\sigma[n]=\begin{cases}1, & n\geq 0\\ 0, & n<0\end{cases}$$

Any given signal x[n] has an odd part and an even part

$$x[n]=x_e[n]+x_o[n]$$

with

$$x_e[n]=\frac{x[n]+x[-n]}{2} \text{ and } x_o[n]=\frac{x[n]-x[-n]}{2}$$

Even signal: $x_e[-n]=x_e[n]$; Odd signal: $x_o[-n]=-x_o[n]$

- ***Energy*** of a discrete-time signal

$$W=\sum_{n=-\infty}^{\infty}|x[n]|^2$$

If the signals has the support $\{N_1, N_{1+1}, \ldots, N_2\}$, the energy is:

$$W=\sum_{n=N_1}^{N_2}|x[n]|^2$$

Power of the discrete-time signal (again support $\{N_1, N_{1+1}, \ldots, N_2\}$)

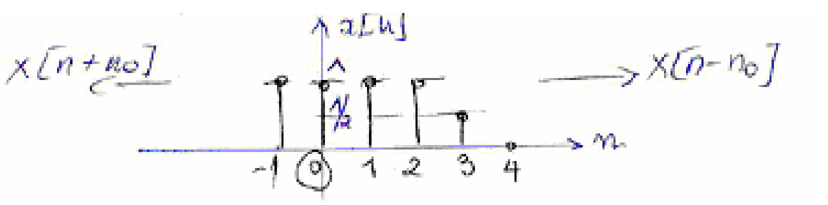
$$P=\frac{W}{N_2-N_1+1}=\frac{1}{N_2-N_1+1}\sum_{n=N_1}^{N_2}|x[n]|^2$$

For periodic signals the power is computed over one period!!!

$$P=\frac{1}{N}\sum_{n\in\langle N\rangle}|x[n]|^2$$

Problems

P4. Consider the signal x[n] sketched below.



Sketch the following signals:

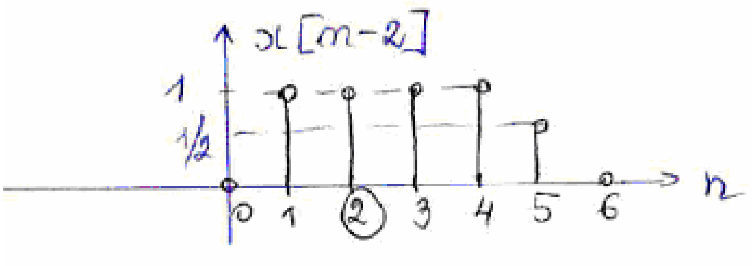
a) $x[n-2]$ d) $x[-n]$ g) $x[n-1]\cdot\delta[n-3]$

b) $x[n+1]$ e) $x[2-n]$

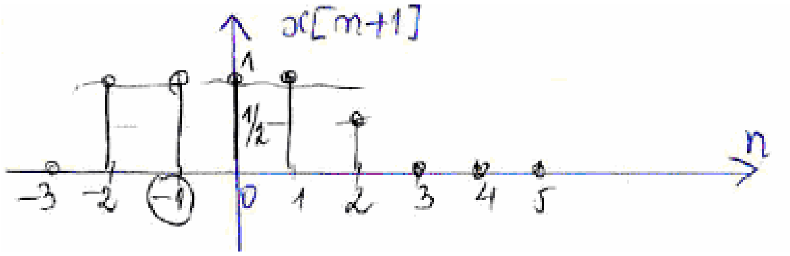
c) $x[2n]$ f) $x[n]\cdot\sigma[1-n]$

Solution.

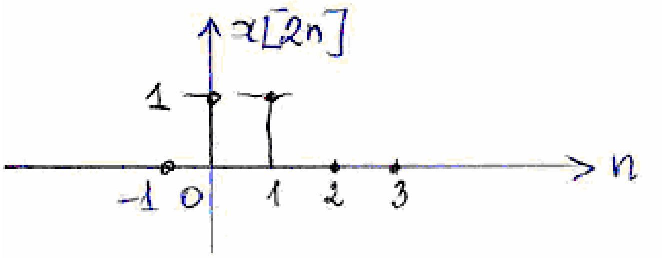
a) $x[n-2]$. The new origin is $n=2$; for $n-2=0$



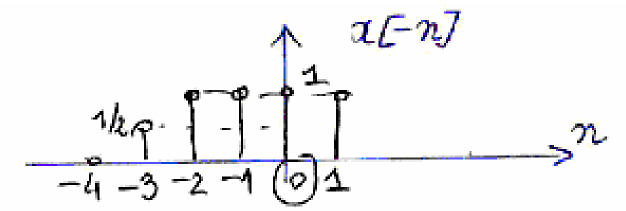
b) $x[n+1]$. The new origin is $n=-1$; for $n+1=0$



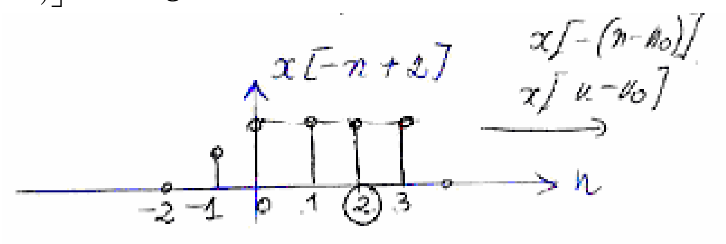
c) $x[2n]$. We have $2\underbrace{n}_{\text{new}}=\underbrace{n'}_{\text{original}}$. Number "n" integer => n' even values: ... -2; 0; 2; 4...



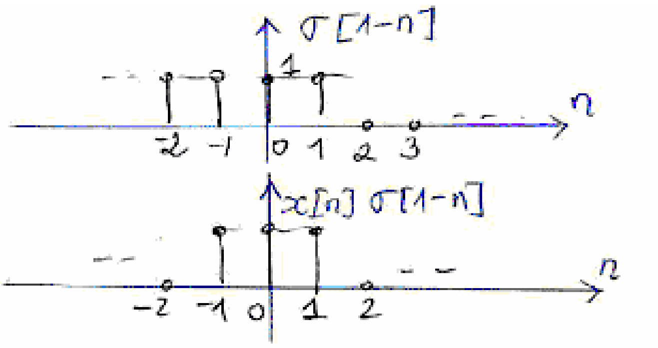
d) $x[-n]$



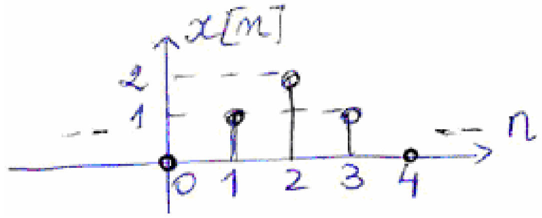
e) $x[2-n]=x[-(n-2)]$. The signal is reversed and then shifted or vice versa.



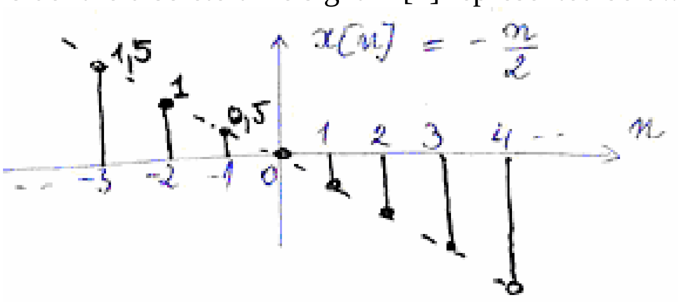
f) $x[n]\cdot\sigma[1-n]$



P5–Homework. Draw the odd and even signals of the x[n] signal represented below:



P6–Homework. Consider the discrete-time signal x[n] represented below.



Sketch the following signals:

a) $x[-n]$ d) $x[2n+1](\sigma[n+2]-\sigma[n-2])$

b) $x[n+1]$ e) $x[n+2](\sigma[n+2]-\sigma[-n])$

c) $x[4n]\cdot\delta[n-2]$

P7. a) If the signal $x[n]$ is odd, prove that: $S=\sum_{n=-\infty}^{\infty}x[n]=\lim_{N\to\infty}\sum_{n=-N}^{N}x[n]=0$

b) If $x_1[n]$ is even and $x_2[n]$ is odd, show that their product $x_1[n]\cdot x_2[n]$ is odd

c) For a finite energy signal $x[n]$, with the even and odd parts $x_e[n]$ and $x_o[n]$, prove that:

$$\sum_{n=-\infty}^{\infty}x^2[n]=\sum_{n=-\infty}^{\infty}x_e^2[n]+\sum_{n=-\infty}^{\infty}x_o^2[n]$$

d) Verify by direct computation the relation from c) for the sketched signal.