Fourier transform

Continuous-time Fourier transform (CTFT)

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t} d\omega \overset{\mathcal{F}}{\longleftrightarrow} X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t} dt.$$

P1. Denote X(ω) – the Fourier transform of the signal x(t). Determine the following values, without computing X(ω).

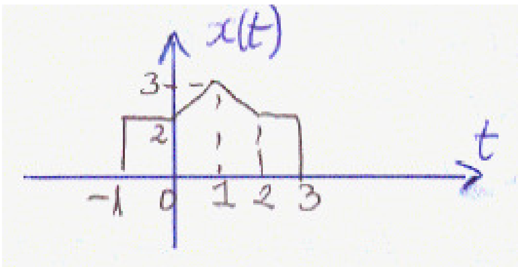
a) X(0)

b) $\int_{-\infty}^{\infty} X(\omega)\, d\omega$

c) $\int_{-\infty}^{\infty} X(\omega)\frac{\sin\omega}{\omega}\, d\omega$

d) $\int_{-\infty}^{\infty} \left|X(\omega)\right|^2 d\omega$

e) inverse Fourier transform for $\mathrm{Re}\{X(\omega)\}$



a) From the definition $X(0) = \int_{-\infty}^{\infty} x(t)\, dt = 9$

b) $\int_{-\infty}^{\infty} X(\omega)\, d\omega = 2\pi x(0) = 4\pi$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t} d\omega \overset{t=0}{=} 2\pi x(0) = 4\pi$$

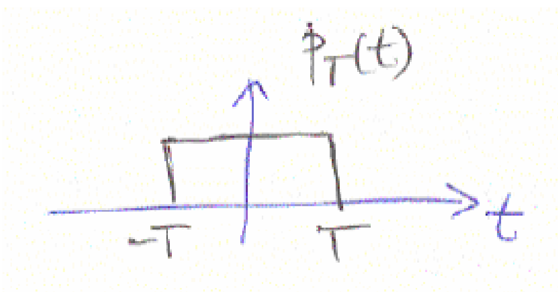
c) $\int_{-\infty}^{\infty} X(\omega)\frac{\sin\omega}{\omega}\, d\omega = \int_{-\infty}^{\infty} Y(\omega)\, d\omega = 2\pi y(0)$

We denote $Y(\omega) = X(\omega)\frac{\sin\omega}{\omega}$

We know that:

$$p_T(t) \leftrightarrow X_p(\omega) = \frac{2\sin\omega\tau}{\omega}$$

(from the table pairs signal-Fourier transform)



$$\Rightarrow Y(\omega) = X(\omega)\,\mathcal{F}\left\{\frac{1}{2} p_1(t)\right\}$$

$$\Rightarrow y(t) = x(t) * \left\{\frac{1}{2} p_1(t)\right\} = \frac{1}{2}\int_{-\infty}^{\infty} x(\tau)\, p_1(t-\tau)\, d\tau$$

$$\Rightarrow y(t)=\frac{1}{2}\int_{t-1}^{t+1} x(\tau)\,d\tau$$

$$y(0)=\frac{1}{2}\int_{-1}^{1} x(\tau)\,d\tau=\frac{9}{4}$$

$$\int_{-\infty}^{\infty} X(\omega)\frac{\sin\omega}{\omega}d\omega=\int_{-\infty}^{\infty} Y(\omega)\,d\omega=2\pi y(0)=2\pi\frac{9}{4}=\frac{9\pi}{2}$$

d) $\int_{-\infty}^{\infty}|X(\omega)|^2 d\omega$ - homework

Apply $\frac{1}{2\pi}\int_{-\infty}^{\infty}|X(\omega)|^2 d\omega=\int_{-\infty}^{\infty}|x(t)|^2 dt$

And from the sketched signal, we have the values of x(t):

$$x(t)=\begin{cases} 0, & t<-1 \\ 2, & -1\le t<0 \\ t+2, & 0\le t<1 \\ 4-t, & 1\le t<2 \\ 2, & 2\le t<3 \\ 0, & t>3 \end{cases}$$

e) The inverse transform of $\operatorname{Re}\{X(\omega)\}$ is $x_p(t)=\frac{x(t)+x(-t)}{2}$ - homework

P2. For a linear invariant time (LTI) system S with input signal x(t) and output signal y(t)

$$x(t)=\left[e^{-t}+e^{-3t}\right]\sigma(t);\quad y(t)=\left[2e^{-t}-2e^{-4t}\right]\sigma(t)$$

a) determine the frequency response of the system
b) determine the impulse response of the system
c) determine the differential equation ; implement the system using integrators, adders and multipliers

Solution.

a) $H(\omega)=?$

$$H(\omega)=\frac{Y(\omega)}{X(\omega)}$$

$$x(t)=\left[e^{-t}+e^{-3t}\right]\sigma(t)\leftrightarrow X(\omega)=\frac{1}{1+j\omega}+\frac{1}{3+j\omega}=\frac{2(2+j\omega)}{(1+j\omega)(3+j\omega)}$$

$$y(t)=\left[2e^{-t}-2e^{-4t}\right]\sigma(t)\leftrightarrow Y(\omega)=\frac{2}{1+j\omega}-\frac{2}{4+j\omega}=\frac{6}{(1+j\omega)(4+j\omega)}$$

$$\Rightarrow H(\omega)=\frac{3(3+j\omega)}{(4+j\omega)(2+j\omega)}$$

b) $h(t)=?\leftrightarrow H(\omega)=\dfrac{3(3+j\omega)}{(4+j\omega)(2+j\omega)}=\dfrac{A}{4+j\omega}+\dfrac{B}{2+j\omega}$

$$A=H(\omega)(4+j\omega)\Big|_{j\omega=-4}=\frac{3(3+j\omega)}{(2+j\omega)}\Bigg|_{j\omega=-4}=\frac{3}{2}$$

$$B=H(\omega)(2+j\omega)\Big|_{j\omega=-2}=\frac{3(3+j\omega)}{(4+j\omega)}\Bigg|_{j\omega=-2}=\frac{3}{2}$$

$$\Rightarrow H(\omega)=\frac{3}{2}\left[\frac{1}{4+j\omega}+\frac{1}{2+j\omega}\right]\leftrightarrow h(t)=\frac{3}{2}\left[e^{-4t}+e^{-2t}\right]\sigma(t)$$

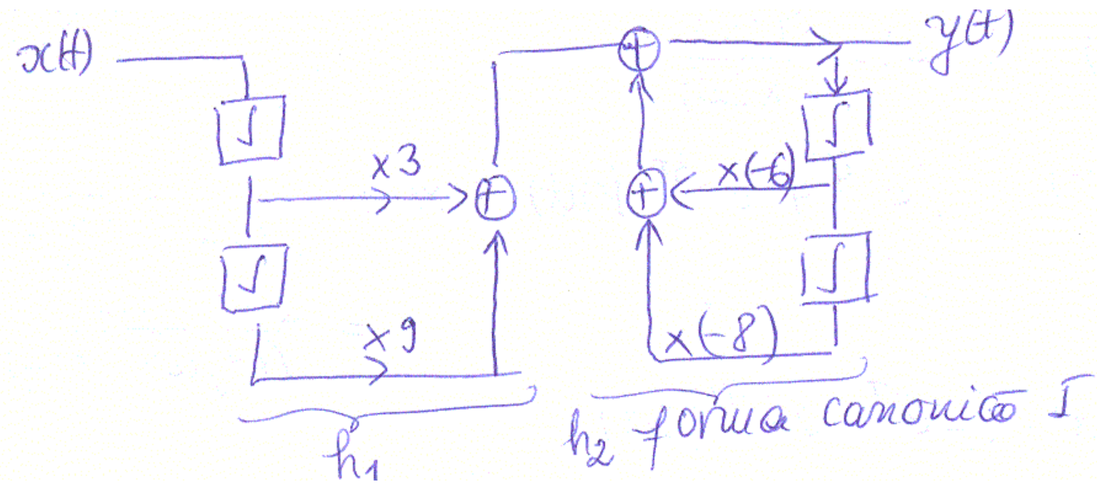
c) $H(\omega)=\dfrac{3(3+j\omega)}{(4+j\omega)(2+j\omega)}=\dfrac{9+3j\omega}{8+6j\omega+(j\omega)^2}=\dfrac{Y(\omega)}{X(\omega)}$

$$\Rightarrow 9X(\omega)+3j\omega X(\omega)=8Y(\omega)+6j\omega Y(\omega)+(j\omega)^2Y(\omega)$$

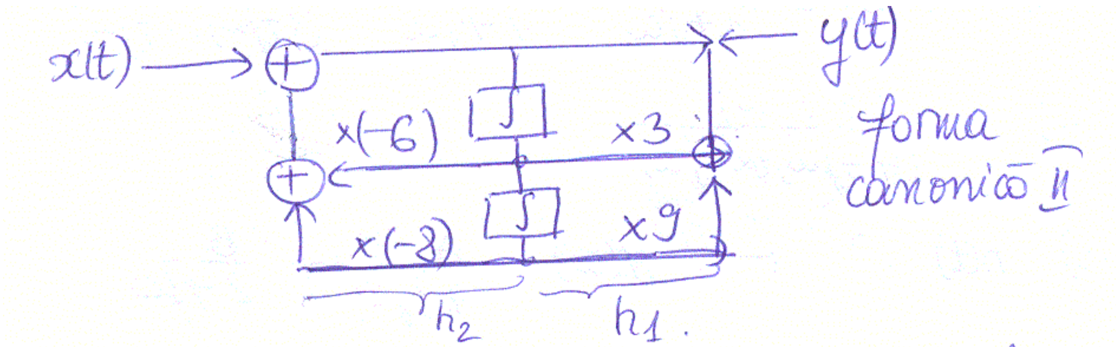
We have as property of Fourier transform: $(j\omega)^n X(\omega)\leftrightarrow \dfrac{d^n x(t)}{dt^n}$

$$\Rightarrow 9x(t)+3x'(t)=8y(t)+6y'(t)+y''(t)$$

$$\Rightarrow 9\int\left[\int x(t)\,dt\right]dt+3\int x(t)\,dt=8\int\left[\int y(t)\,dt\right]dt+6\int y(t)\,dt+y(t)$$



Canonical form I



Canonical form II

P3. The signal $x(t)=\cos\alpha t+\cos 3\alpha t$ is connected to the input of the system with the following impulse response (three cases):

$$h_a(t)=\frac{\sin(2\alpha t)}{\frac{\alpha}{2}t};\qquad h_b(t)=\frac{\sin(2\alpha t)\sin(4\alpha t)}{\frac{\alpha}{2}t^2};\qquad h_c(t)=\frac{\sin(2\alpha t)\cos(4\alpha t)}{\frac{\alpha}{2}t}$$

a) Determine the frequency response of the system in the three cases. Sketch the corresponding magnitude spectra.
b) Determine the response of the three systems to x(t)
c) If x(t) would has a useful component and a noisy component, which of the three systems would you pick, assuming the disturbance is $\cos(\alpha t)$? What if it's $\cos(3\alpha t)$?

Solution

a) $H_a(\omega),H_b(\omega),H_c(\omega)=?$

From the table pairs signals-Fourier transform we have,

$$\frac{\sin(\omega_o t)}{\pi t}\overset{\mathcal{F}}{\leftrightarrow}p_{\omega_0}(\omega)=\begin{cases}1,|\omega|<\omega_0\\0,|\omega|>\omega_0\end{cases}$$

$$h_a(t)=\frac{2\pi}{\alpha}\frac{\sin(2\alpha t)}{\pi t}\overset{\mathcal{F}}{\leftrightarrow}\frac{2\pi}{\alpha}p_{2\alpha}(\omega)=H_a(\omega)$$

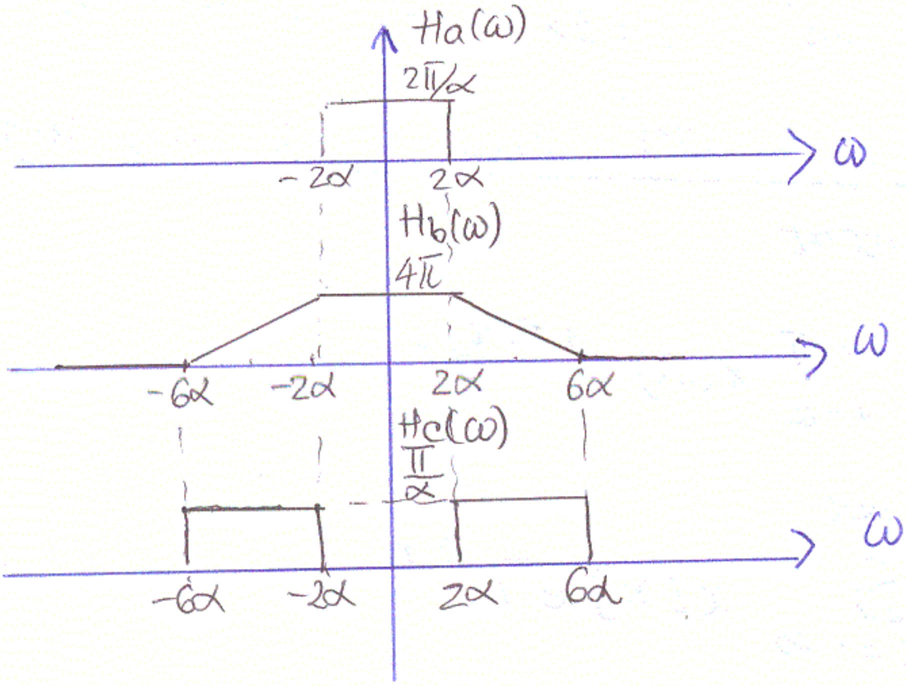
$$h_b(t)=\frac{2\pi^2}{\alpha}\frac{\sin(2\alpha t)}{\pi t}\frac{\sin(4\alpha t)}{\pi t}\leftrightarrow\frac{2\pi^2}{\alpha}\cdot\frac{1}{2\pi}p_{2\alpha}(\omega)*p_{4\alpha}(\omega)$$

$$=\frac{\pi}{\alpha}p_{2\alpha}(\omega)*p_{4\alpha}(\omega)=H_b(\omega)$$

$$h_c(t)=\frac{\sin(2\alpha t)\cos(4\alpha t)}{\frac{\alpha}{2}t}=\frac{\sin(6\alpha t)-\sin(2\alpha t)}{2\frac{\alpha}{2}t}$$

$$\sin a\cos b=\frac{1}{2}\left[\sin(a+b)+\sin(a-b)\right]$$

$$h_c(t)=\frac{\pi}{\alpha}\left[\frac{\sin(6\alpha t)}{\pi t}-\frac{\sin(2\alpha t)}{\pi t}\right]\leftrightarrow H_c(\omega)=\frac{\pi}{\alpha}\left[p_{6\alpha}(\omega)-p_{2\alpha}(\omega)\right]$$



$$H_b(c)=\frac{\pi}{\alpha}\cdot\begin{cases}0, & t<-6\alpha\\ 6\alpha+t, & -6\alpha\le t<-2\alpha\\ 4\alpha, & -2\alpha\le t<2\alpha\\ 6\alpha-t, & 2\alpha\le t<6\alpha\\ 0, & t\ge 6\alpha\end{cases}$$

b) $y_a(t), y_b(t), y_c(t)=?$

For $x(t)=\cos(\omega_0 t)\rightarrow y(t)=|H(\omega_0)|\cos\left(\omega_0 t+\arg\{H(\omega_0)\}\right)$

$$x(t)=\cos\alpha t+\cos 3\alpha t$$

$$y(t)=|H(\alpha)|\cos\left(\alpha t+\arg\{H(\alpha t)\}\right)+|H(3\alpha)|\cos\left(3\alpha t+\arg\{H(3\alpha t)\}\right)$$

For $H_a(\omega)$, we have the output signal

$$y_a(t) = |H_a(\alpha)|\cos(\alpha t + \arg\{H_a(\alpha t)\}) + |H_a(3\alpha)|\cos(3\alpha t + \arg\{H_a(3\alpha t)\})$$

$$\Rightarrow y_a(t) = \frac{2\pi}{\alpha}\cos(\alpha t)$$

In the same manner, we have

$$y_b(t) = 4\pi\cos(\alpha t) + 3\pi\cos(3\alpha t)$$

$$y_c(t) = \frac{\pi}{\alpha}\cos(3\alpha t)$$

Discrete-time Fourier transform (DTFT)

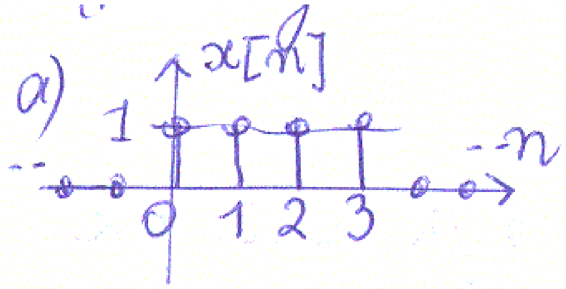
An aperiodic discrete-time signal can be decomposed in

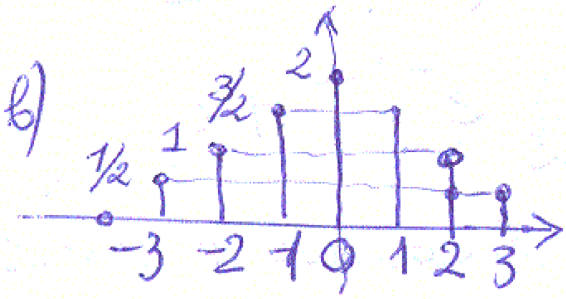
$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\Omega)e^{j\Omega n}d\Omega$$

The Fourier transform

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$$

P1. Determine the fourier transform for the following discrete-time signals





c) $x_3[n] = \left(\frac{1}{4}\right)^n \sigma[n+2]$

Solution.

By definition, the DTFT is $X(\Omega)=\sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$

a)

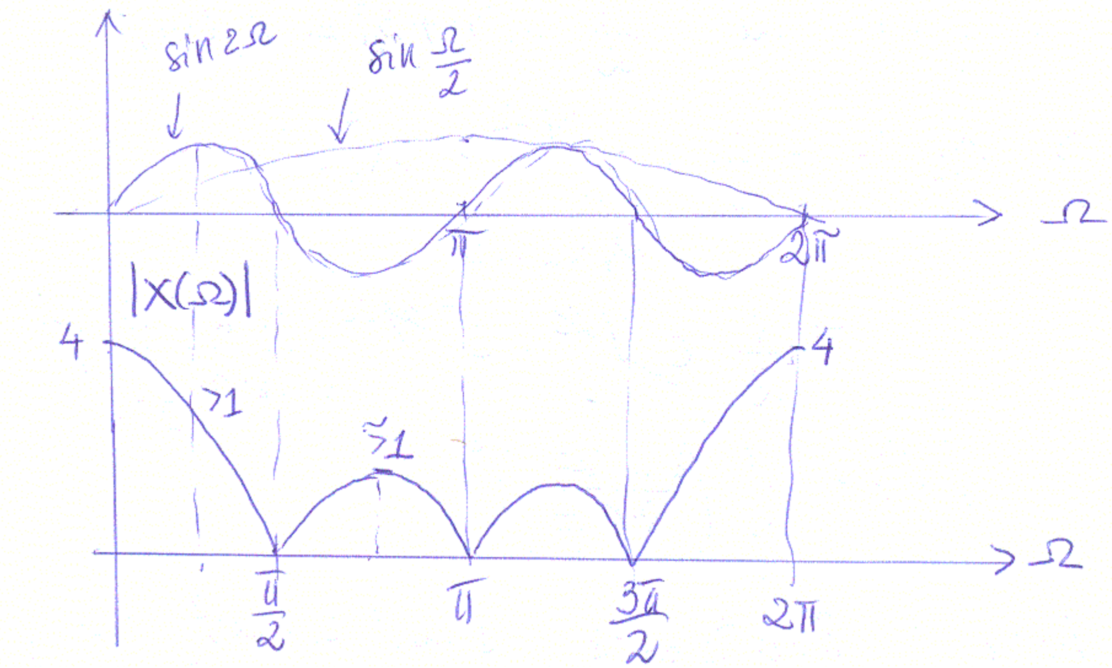
$$X(\Omega)=1\cdot e^{0}+1\cdot e^{-j\Omega}+1\cdot e^{-2j\Omega}+1\cdot e^{-3j\Omega}=\frac{1-e^{-4j\Omega}}{1-e^{-j\Omega}}=\frac{e^{-2j\Omega}\left(e^{2j\Omega}-e^{-2j\Omega}\right)}{e^{-j\frac{\Omega}{2}}\left(e^{j\frac{\Omega}{2}}-e^{-j\frac{\Omega}{2}}\right)}$$

$$X(\Omega)=e^{-3j\frac{\Omega}{2}}\frac{2j\cdot\sin(2\Omega)}{2j\cdot\sin\left(\frac{\Omega}{2}\right)}=e^{-3j\frac{\Omega}{2}}\frac{\sin(2\Omega)}{\sin\left(\frac{\Omega}{2}\right)}$$

$\rightarrow T_1=\pi$

$\rightarrow T_2=4\pi$

$$|X(\Omega)|=\frac{|\sin(2\Omega)|}{\left|\sin\left(\frac{\Omega}{2}\right)\right|}$$



b)

$$X_2(\Omega)=\frac{1}{2}\cdot e^{3j\Omega}+\frac{1}{2}\cdot e^{-3j\Omega}+1\cdot e^{2j\Omega}+1\cdot e^{-2j\Omega}+\frac{3}{2}\cdot e^{j\Omega}+\frac{3}{2}\cdot e^{-j\Omega}+2$$

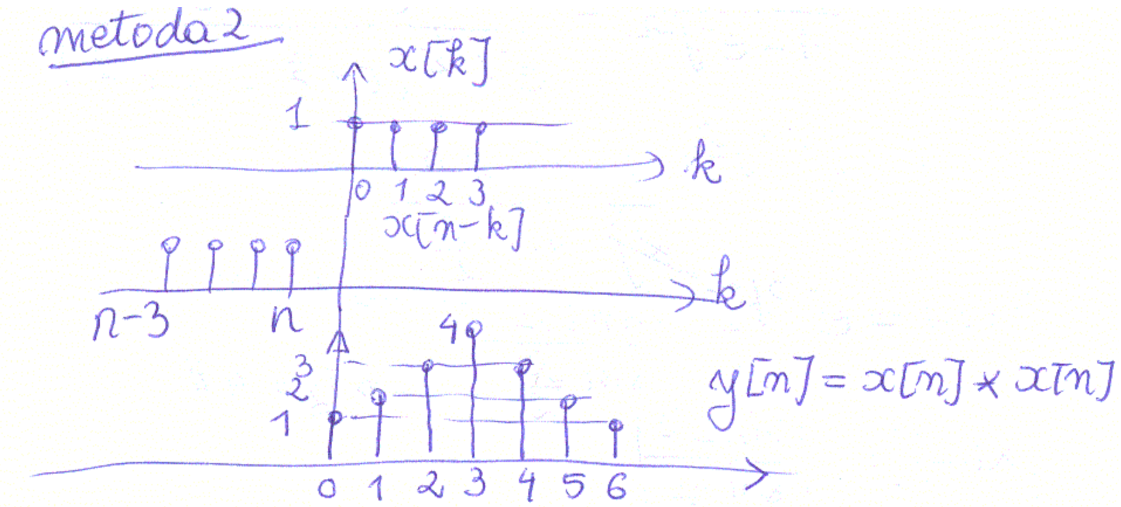
$$X_2(\Omega)=\frac{1}{2}\cdot 2\cos(3\Omega)+2\cos(2\Omega)+\frac{3}{2}\cdot 2\cos\Omega+2$$

$$=4\cos^3\Omega-3\cos\Omega+2\left(2\cos^2\Omega-1\right)+3\cos\Omega+2$$

$$X_2(\Omega)=4\cos^3\Omega+4\cos^2\Omega=4\cos^2\Omega(\cos\Omega+1)$$

$$X_2(\Omega) = 4\cos^2\Omega \cdot 2\cos^2\frac{\Omega}{2} = 8\cos^2\Omega \cdot \cos^2\frac{\Omega}{2}$$

Second method:



Notice that

$$\frac{1}{2}y[n+3] = x_2[n]$$

$$X_2(\Omega) = \frac{1}{2}\underbrace{e^{3j\Omega}}_{\text{shifting}}Y(\Omega) = \frac{1}{2}e^{3j\Omega}e^{-3j\Omega}\left(\frac{\sin 2\Omega}{\sin\frac{\Omega}{2}}\right)^2$$

$$X_2(\Omega) = \frac{1}{2}\left(\frac{\sin 2\Omega}{\sin\frac{\Omega}{2}}\right)^2$$

We can verify that the two results are the same:

$$8\cos^2\Omega \cdot \cos^2\frac{\Omega}{2} = \frac{8\cos^2\Omega \cdot \cos^2\frac{\Omega}{2} \cdot \sin^2\frac{\Omega}{2}}{\sin^2\frac{\Omega}{2}} = \frac{2\cos^2\Omega \cdot \sin^2\Omega}{\sin^2\frac{\Omega}{2}} = \frac{1}{2}\frac{\sin^2(2\Omega)}{\sin^2\frac{\Omega}{2}}$$

c)

$$x_3[n] = \left(\frac{1}{4}\right)^n \sigma[n+2]$$

Direct computation

$$X_3(\Omega) = \sum_{n=-\infty}^{\infty} x_3[n]e^{-j\Omega n}$$

$$X_3(\Omega)=\left(\frac{1}{4}\right)^{-2}\cdot e^{2j\Omega}+\left(\frac{1}{4}\right)^{-1}\cdot e^{j\Omega}+\left(\frac{1}{4}\right)^{0}\cdot 1+\left(\frac{1}{4}\right)^{1}\cdot e^{-j\Omega}+\left(\frac{1}{4}\right)^{2}\cdot e^{-2j\Omega}+\left(\frac{1}{4}\right)^{3}\cdot e^{-3j\Omega}+\ldots$$

$$X_3(\Omega)=\left(\frac{1}{4}\right)^{-2}\cdot e^{2j\Omega}\left[1+\left(\frac{1}{4}\right)\cdot e^{-j\Omega}+\left(\frac{1}{4}\right)^{2}\cdot e^{-2j\Omega}+\ldots\right]$$

$$=\left(\frac{1}{4}\right)^{-2}\cdot e^{2j\Omega}\frac{1}{1-\frac{1}{4}e^{-j\Omega}}=\frac{16e^{2j\Omega}}{1-\frac{1}{4}e^{-j\Omega}}$$

From the table Pairs signals- Fourier transform

$$a^n\sigma[n]\leftrightarrow\frac{1}{1-ae^{-j\Omega}}$$

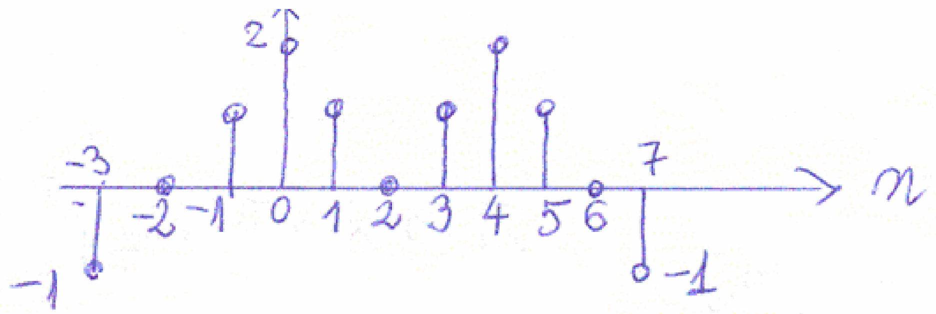
$$y[n]=x_3[n-2]=\left(\frac{1}{4}\right)^{n-2}\sigma[n]=16\left(\frac{1}{4}\right)^{n}\sigma[n]$$

$$Y(\Omega)=16\frac{1}{1-\frac{1}{4}e^{-j\Omega}}=e^{-2j\Omega}X_3(\Omega)$$

$$\Rightarrow X_3(\Omega)=e^{2j\Omega}Y(\Omega)=\frac{16e^{2j\Omega}}{1-\frac{1}{4}e^{-j\Omega}}$$

$$\left|X_3(\Omega)\right|=\frac{16}{\sqrt{\left(1-\frac{1}{4}\cos\Omega\right)^2+\left(\frac{1}{4}\sin\Omega\right)^2}}=\frac{16}{\sqrt{\frac{17}{16}-\frac{1}{2}\cos\Omega}}$$

P2. We have the signal x(t)



a) X(0)
b) $\arg\{X(\Omega)\}$
c) $\int_{-\pi}^{\pi}X(\Omega)\,d\Omega$

d) X(π)

e) $\int_{-\pi}^{\pi} \left| X(\Omega) \right|^2 d\Omega$

f) $\int_{2\pi} \left| \frac{dX(\Omega)}{d\Omega} \right|^2 d\Omega$

g) inverse Fourier transform for $\mathrm{Re}\{X(\Omega)\}$

Solution.

a) $X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n} \Rightarrow X(0) = \sum_{n=-\infty}^{\infty} x[n] = 6$

b) $\arg\{X(\Omega)\}$

Let $y[n] = x[n+2]$ - symmetrical – even signal $\Rightarrow Y(\Omega) \in R$

But $X(\Omega) = e^{-2j\Omega} Y(\Omega)$

$\arg\{X(\Omega)\} = -2\Omega + \arg\{Y(\Omega)\} = -2\Omega$

c) $I = \int_{-\pi}^{\pi} X(\Omega) d\Omega = ?$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega) e^{j\Omega n} d\Omega$$

$I = 2\pi x[0] = 4\pi$

d) $X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n} \Rightarrow X(\pi) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\pi n} = 2$, because $e^{-j\pi n} = (-1)^n$

e) $I = \int_{-\pi}^{\pi} \left| X(\Omega) \right|^2 d\Omega = 2\pi(1+1+4+1) \cdot 2$ because we have $\sum \left| x[n] \right|^2 = \frac{1}{2\pi} I$

f) $\int_{2\pi} \left| \frac{dX(\Omega)}{d\Omega} \right|^2 d\Omega$ -homework

Properties: $nx[n] \leftrightarrow j\frac{dX(\Omega)}{d\Omega} \Rightarrow \sum (nx[n])^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \frac{dX(\Omega)}{d\Omega} \right|^2 d\Omega$

g) The inverse Fourier transform of $\mathrm{Re}\{X(\Omega)\}$ is $x_p[n] = \frac{x[n] + x[-n]}{2}$ -homework