3. Continuous and discrete time Fourier series

Quick overview:

Continuous time Fourier series

The signal x(t) can be decomposed into a Fourier series

$$x(t) = \sum_k c_k e^{jk\omega_0 t} \qquad \omega_0 = \frac{2\pi}{T}$$

The Fourier transform is defined by

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

where x(t) is the c.t. signal.

Then the coefficients of the exponential Fourier series are

$$\{c_k\} = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt$$

$$c_k = \frac{1}{T} X(k\omega_0)$$

Discrete time Fourier series

The discrete time signal x[n] can be decomposed into a Fourier series:

$$x[n] = \sum_{k=0}^{N-1} c_k e^{jk\Omega_0 n} \qquad \Omega_0 = \frac{2\pi}{N}$$

The Fourier transform:

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}$$

Coefficients of the Fourier series

$$\{c_k\} = \frac{1}{N} \sum_n x[n] e^{-jk\Omega_0 n}$$

$$c_k = \frac{1}{N} X(k\Omega_0)$$

Where $|c_k|$ -magnitude spectrum

$\arg\{c_k\}$ -phase spectrum

$|c_k|^2$ -power spectrum

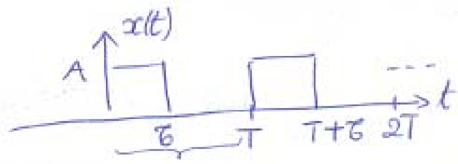
Continuous-time Fourier series. Problems.

1) Determine and sketch the magnitude spectra of the following

a) $x(t) = A \cos \omega_0 t$

b) $x(t) = A \sin \omega_0 t$

c)



d) $x(t) = \cos \frac{\pi (t-3)}{4}$

e) $x(t) = \cos 2t + \sin 3t.$

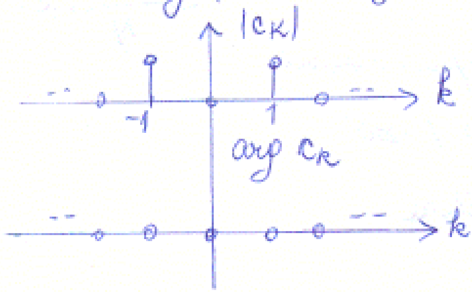
Solution.

a) $x(t) = \frac{A}{2}\left[e^{j\omega_0 t} + e^{-j\omega_0 t}\right]$

$k = 1 \rightarrow c_1 = \frac{A}{2}$ $\qquad$ $k = -1 \rightarrow c_{-1} = \frac{A}{2}$

$|c_1| = |c_{-1}| = \frac{A}{2}$

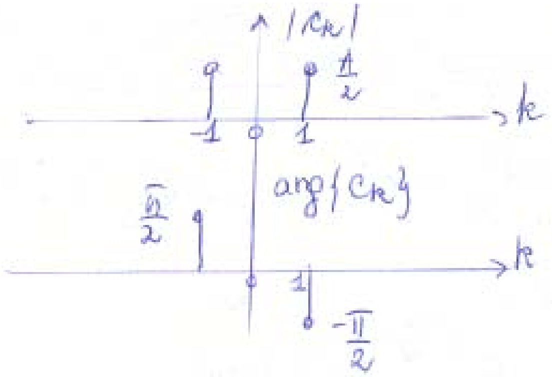
$\arg\{c_1\} = \arg\{c_{-1}\} = 0.$



b) $x(t) = \frac{A}{2j}\left[e^{j\omega_0 t} - e^{-j\omega_0 t}\right]$

$c_1 = -\frac{A}{2} j \qquad |c_1| = \frac{A}{2} \qquad \arg\{c_1\} = -\frac{\pi}{2}$

$c_{-1} = \frac{A}{2} j \qquad |c_{-1}| = \frac{A}{2} \qquad \arg\{c_{-1}\} = \frac{\pi}{2}$



c) $x(t) = \begin{cases} A, & 0 \le t \le \tau \\ 0 & \tau < t < T \end{cases}$

$$c_k = \frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t}\,dt = \frac{1}{T}\int_0^{\tau} A\, e^{-jk\omega_0 t}\,dt$$

$$c_k = \frac{A}{T}\;\frac{e^{-jk\omega_0 t}}{(-jk\omega_0)}\Big|_0^{\tau} = \frac{A}{T}\cdot\frac{e^{-jk\omega_0\tau}-1}{-jk\omega_0}$$

$$c_k = \frac{A}{T}\;\frac{e^{-jk\frac{\omega_0\tau}{2}}}{-jk\omega_0}\left(e^{-jk\frac{\omega_0\tau}{2}} - e^{jk\frac{\omega_0\tau}{2}}\right)$$

$$\frac{2A\cdot e^{-jk\frac{\omega_0\tau}{2}}}{k\cdot 2\pi}\cdot\frac{e^{jk\frac{\omega_0\tau}{2}} - e^{-jk\frac{\omega_0\tau}{2}}}{2j}$$

$$c_k = \frac{A}{k\pi}\,e^{-jk\frac{\omega_0\tau}{2}}\cdot\sin\left(\frac{k\omega_0\tau}{2}\right) \qquad k \neq 0$$

$$|c_k| = \frac{A}{k\pi}\left|\sin\frac{k\omega_0\tau}{2}\right|$$

$$\arg\{c_k\} = \operatorname{arctg}\frac{Im(c_k)}{Re(c_k)} = \operatorname{arctg}\left(-\frac{\sin\left(\frac{k\omega_0\tau}{2}\right)}{\cos\left(\frac{k\omega_0\tau}{2}\right)}\right)$$

$$\arg\{c_k\} = -\frac{k\omega_0\tau}{2}.$$

$$k=0: \quad c_0 = \frac{1}{T}\int_T x(t)\,dt = \frac{A}{T}\int_0^{\tau} dt = \frac{A\tau}{T}$$

$$\begin{cases} |c_0| = \dfrac{A\tau}{T} \\ \arg\{c_0\} = 0 \end{cases}$$

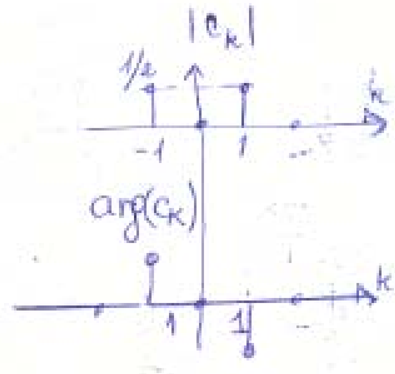
d) $$x(t) = \cos\left[\frac{\pi(t-3)}{4}\right] = \frac{e^{j\frac{\pi(t-3)}{4}} + e^{-j\frac{\pi(t-3)}{4}}}{2}$$

$$= \underbrace{\frac{1}{2}e^{-j\frac{3\pi}{4}}}_{c_{+1}} e^{j\frac{\pi}{4}t} + \underbrace{\frac{1}{2}e^{j\frac{3\pi}{4}}}_{c_{-1}} e^{-j\frac{\pi}{4}t}$$

$$e^{j\frac{2\pi}{T}t} = e^{j\frac{\pi}{4}t} \Rightarrow T = 8.$$

$$\begin{cases} |c_{+1}| = \left|\frac{1}{2}e^{-j\frac{3\pi}{4}}\right| = \frac{1}{2} \\ |c_{-1}| = \left|\frac{1}{2}e^{j\frac{3\pi}{4}}\right| = \frac{1}{2} \end{cases}$$

$$\begin{cases} \arg c_1 = -\frac{3\pi}{4} \\ \arg c_{-1} = \frac{3\pi}{4} \end{cases}$$



e)

$$x(t) = \cos 2t + \sin 3t \qquad T_1 = \pi,\ T_2 = \frac{2\pi}{3}$$

$$= \underbrace{\frac{1}{2}}_{c_2}e^{j2t} + \underbrace{\frac{1}{2}}_{c_{-2}}e^{-j2t} + \underbrace{\frac{1}{2j}}_{c_3}e^{j3t} - \underbrace{\frac{1}{2j}}_{c_{-3}}e^{-j3t}$$

$$x(t+T) = x(t)$$

$T = 2\pi = \text{cmmmc}(T_1, T_2)$

$c_2 = c_{-2} = \frac{1}{2}$

$c_3 = -\frac{1}{2} j$

$c_{-3} = \frac{1}{2} j$

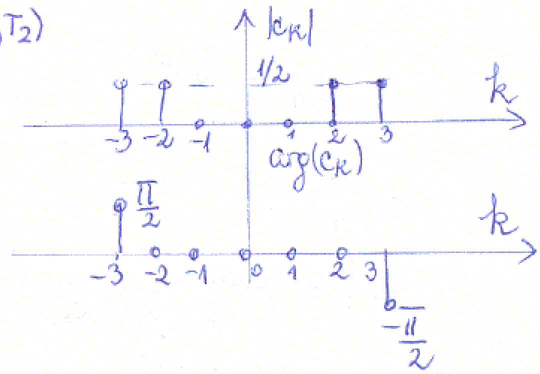
$|c_2| = |c_{-2}| = \frac{1}{2}$

$|c_3| = |c_{-3}| = \frac{1}{2}$

$\arg\{c_2\} = \arg\{c_{-2}\} = 0$

$\arg c_3 = -\frac{\pi}{2}$

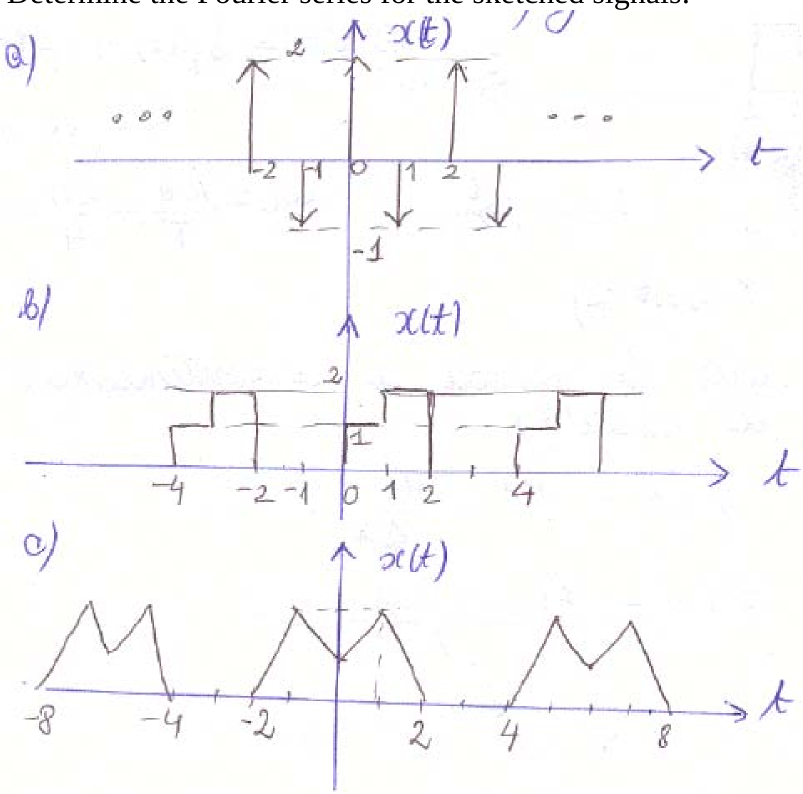
$\arg c_{-3} = \frac{\pi}{2}$



Homework. $x(t) = e^{-2t}$, for the period T=4 and $t \in (-2, 2)$. Fourier coefficients are

$$c_k = \frac{1}{T} \int_{-2}^{2} x(t)\, e^{-jk\omega_0 t}\, dt$$

2) Determine the Fourier series for the sketched signals:



$$a)\quad T=2 \Rightarrow \omega_0 = \frac{2\pi}{T} = \pi$$
$$x(t) = 2\,\delta_T(t) - \delta_T(t-1)$$

Solution.
From the pairs signal – Fourier transform we have

$$\delta_T(t) \longleftrightarrow c_k = \frac{1}{T}$$

From the properties of the exponential Fourier series

$$x(t-t_0) \longleftrightarrow c_k = e^{-jk\omega_0 t_0} c_k^x$$

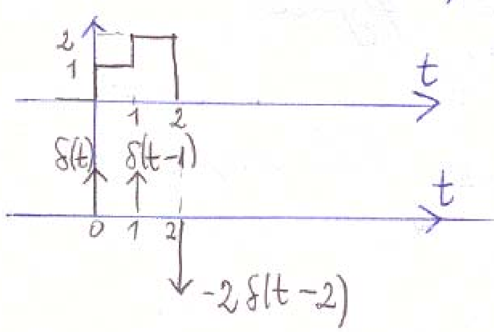
The Fourier series for the signal x(t) then becomes:

$$c_k^x = 2\cdot\frac{1}{T} - e^{-jk\omega_0\cdot 1}\cdot\frac{1}{T}$$
$$= \frac{1}{2}\left(2 - e^{-jk\pi}\right) = \frac{1}{2}\left[2-(-1)^k\right].$$

Remember that

$$e^{-jk\pi} = \cos k\pi - j\sin k\pi.$$

b) We obtain the nth order derivative of the signal, until this is a sum of unit impulses (Dirac); we take only the restricted signal for T=4.



$$\Rightarrow x'(t) = \delta_T(t) + \delta_T(t-1) - 2\,\delta_T(t-2)$$
$$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{4} = \frac{\pi}{2}$$

The 1st order derivative in discontinuity points is

$$f'(x) = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$. For example, at 0 we have,

$$f'(0) = \frac{f(1)-f(0)}{1} = 1.$$

$$c_k^{x'} = jk\omega_0\, c_k^x, \quad k \neq 0$$
$$\Rightarrow c_k^x = \frac{c_k^{x'}}{jk\omega_0}.$$

$$c_k^{x'} = \frac{1}{T} + e^{-jk\omega_0 \cdot 1}\frac{1}{T} - 2e^{-jk\omega_0 2}\cdot\frac{1}{T}$$

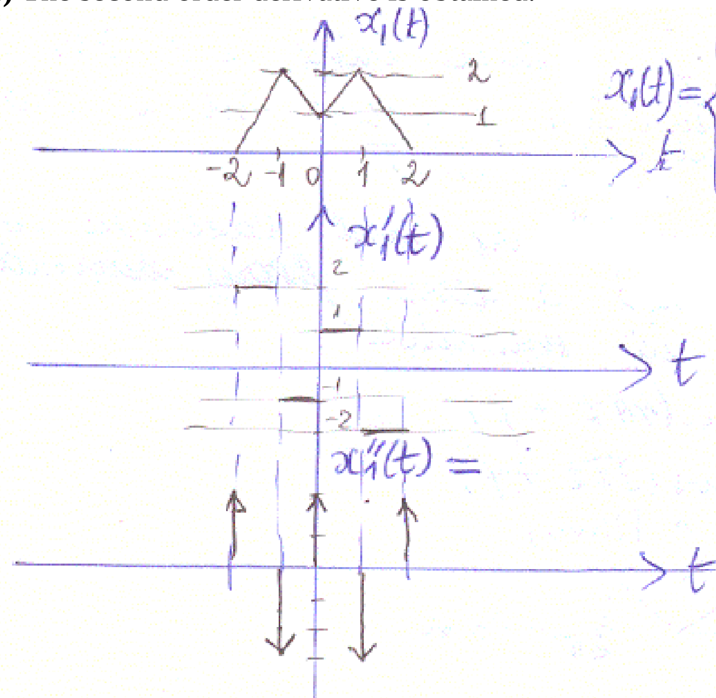
$$c_k^{x'} = \frac{1}{4}\left(1 + e^{-jk\frac{\pi}{2}} - 2e^{-jk\pi}\right)$$

$$\boxed{c_k^{x} = \frac{1}{jk\frac{\pi}{2}}\cdot\frac{1}{4}\left(1 + e^{-jk\frac{\pi}{2}} - 2e^{-jk\pi}\right) \quad k\neq 0.}$$

$k=0$

$$c_0 = \frac{1}{T}\int_T x(t)\,dt = \frac{3}{4} \Rightarrow \boxed{c_0^x = \frac{3}{4}}$$

c) The second order derivative is obtained.



$$x_1(t) = \begin{cases} 2(t+2), & t\in(-2,-1) \\ 1-t, & t\in(-1,0) \\ t+1, & t\in(0,1) \\ 2(2-t) & t\in(1,2) \end{cases}$$

$$x_1''(t) = 2\delta(t+2) - 3\delta(t+1) + 2\delta(t) - 3\delta(t-1) + 2\delta(t-2)$$

$$\Rightarrow x''(t) = 2\delta_T(t+2) - 3\delta_T(t+1) + 2\delta_T(t) - 3\delta_T(t-1) + 2\delta_T(t-2)$$

$$c_k^{x''} = (jk\omega_0)^2 c_k^x \quad , k\neq 0$$

$$c_k^x = \frac{c_k^{x''}}{(jk\omega_0)^2} \qquad \omega_0 = \frac{2\pi}{T} = \frac{\pi}{3}, \quad T=6.$$

$$c_k^{x''} = 2\cdot e^{-jk\frac{\pi}{3}(-2)}\cdot\frac{1}{T} - 3e^{-jk\frac{\pi}{3}(-1)}\cdot\frac{1}{T} + 2\cdot\frac{1}{T} - 3\cdot e^{-jk\frac{\pi}{3}}\frac{1}{T} + 2e^{-jk\frac{\pi}{3}2}\cdot\frac{1}{T}$$

$$c_k^{x''} = \frac{1}{6}\left[2e^{jk\frac{2\pi}{3}} - 3e^{jk\frac{\pi}{3}} + 2 - 3e^{-jk\frac{\pi}{3}} + 2e^{-jk\frac{2\pi}{3}}\right]$$

$$c_k^{x} = -\frac{9c_k^{x''}}{k^2\cdot\pi^2} \qquad k\neq 0$$

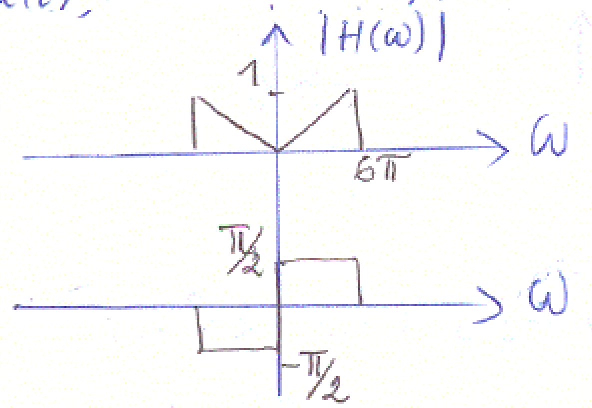
$$k=0 \qquad c_0 = \frac{1}{T}\int_T x(t)\,dt = \frac{5}{6}$$

3)

a)Determine the connection between the Fourier coefficients of the input periodic signal and the output periodic signal, from a linear time-invariant system LTI.

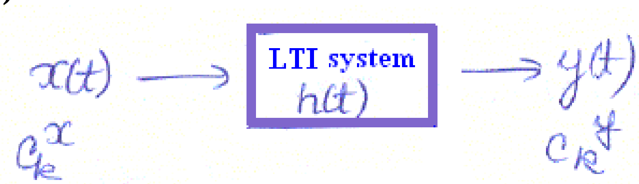
b)Determine the response of a filter (magnitude and phase spectra below) to the following signals:

$$x(t) = \cos(4\pi t + \varphi)$$
$$x(t) = \cos(8\pi t + \varphi)$$



Solution.

a)



$$y(t) = \sum_k c_k^y e^{jk\omega_0 t}$$

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau) \left(\sum_k c_k^x e^{jk\omega_0(t-\tau)} \right) d\tau$$

$$= \sum_k c_k^x \cdot e^{jk\omega_0 t} \underbrace{\int_{-\infty}^{\infty} h(\tau) e^{-jk\omega_0 \tau} d\tau}_{H(k\omega_0)}, \quad k>0$$

$$= \sum c_k^x H(\omega_0) e^{jk\omega_0 t}$$

b) $x(t) = \cos(4\pi t + \varphi)$

$$= \frac{e^{j(4\pi t+\varphi)} + e^{-j(4\pi t+\varphi)}}{2}$$

$$= \underbrace{\frac{e^{j\varphi}}{2}}_{c_1} e^{j4\pi t} + \underbrace{\frac{e^{-j\varphi}}{2}}_{c_{-1}} e^{-j4\pi t}$$

$4\pi = \frac{2\pi}{T} \Rightarrow T = \frac{1}{2}$ $\qquad \omega_0 = \frac{2\pi}{T} = 4\pi$

$$\begin{cases} c_1^x = \dfrac{e^{j\varphi}}{2} \\ c_{-1}^x = \dfrac{e^{-j\varphi}}{2} \end{cases}$$

$$\begin{matrix} |H(4\pi)| = x \ldots 4\pi \\ |H(6\pi)| = 1 \ldots 6\pi \end{matrix} \Rightarrow |H(4\pi)| = \frac{4\pi}{6\pi} = \frac{2}{3}$$

$$c_k^y = c_k^x \cdot H(k\omega_0)$$

$$|c_k^y| = |c_k^x| \cdot |H(k4\pi)|$$

$$\begin{cases} |c_1^y| = \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3} \\ |c_{-1}^y| = \frac{1}{2} \cdot |H(-4\pi)| = \frac{1}{3} \end{cases}$$

$$\begin{cases} \arg c_1^y = \arg\{c_1^x\} + \arg\{H(4\pi)\} = \varphi + \frac{\pi}{2} \\ \arg c_{-1}^y = \arg\{c_{-1}^x\} + \arg\{H(-4\pi)\} = -\varphi - \frac{\pi}{2} \end{cases}$$

Homework:

$$x(t) = \cos(8\pi t + \varphi)$$

Discrete-time Fourier series. Problems

1) Determine and sketch the magnitude, phase and power spectra for the following:

a) $x[n] = \cos\frac{\pi}{4}(n-3)$

b) $x[n] = \cos\left(\frac{\pi}{2}n\right) + \cos\left(\frac{2\pi}{5}n\right)$

c) $x[n] = e^{-2n}$ pt $n \in \{-2, -1, 0, 1\}$

a) If x[n] is periodic with period N, the signal can be written as

$$x[n] = \sum_{k=0}^{N-1} c_k e^{jk\frac{2\pi}{N}n} \qquad n = \overline{0, N-1}$$

with the Fourier coefficients

$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n] e^{-jk\frac{2\pi}{N}n} \qquad k = \overline{0, N-1}$$

Then the signal becomes:

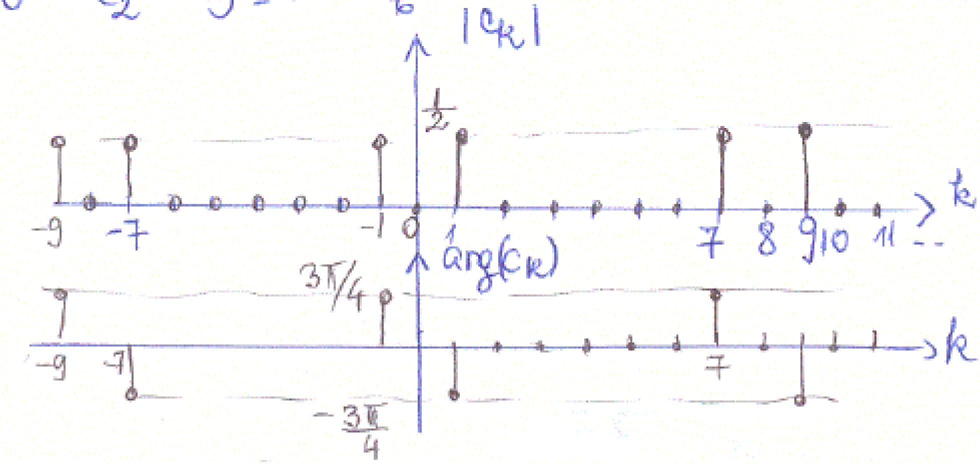
$$x[n] = \cos\frac{\pi}{4}(n-3) = \frac{1}{2}e^{j\frac{\pi}{4}(n-3)} + \frac{1}{2}e^{-j\frac{\pi}{4}(n-3)}$$

$$= \underbrace{\frac{1}{2}e^{-j\frac{3\pi}{4}}}\, e^{j\frac{\pi}{4}n} + \underbrace{\frac{1}{2}e^{j\frac{3\pi}{4}}}\, e^{-j\frac{\pi}{4}n}$$

$$N = 8 \Rightarrow \sum_{k=0}^{7} c_k e^{jk\frac{2\pi}{8}n} = \sum_{k=0}^{7} c_k e^{jk\frac{\pi}{4}n}$$

$$\begin{cases} c_1 = \frac{1}{2} e^{-j\frac{3\pi}{4}} & |c_1| = \frac{1}{2}, \ \arg\{c_1\} = -\frac{3\pi}{4} \\ c_{-1} = c_7 = \frac{1}{2} e^{j\frac{3\pi}{4}} & |c_7| = \frac{1}{2}, \ \arg\{c_7\} = \frac{3\pi}{4} \end{cases}$$

$$c_0 = c_2 = c_3 = \dots = c_6 = 0$$



b) $x[n] = \cos\left(\frac{\pi}{2} n\right) + \cos\left(\frac{2\pi}{5} n\right)$

$x_1[n] = \cos\left(\frac{2\pi}{N_1} n\right) \qquad N_1 - \text{per} \qquad N_1 = 4$

$x_2[n] = \cos\left(\frac{2\pi}{N_2} n\right) \qquad N_2 - \text{per}. \qquad N_2 = 5$

$x[n] = x_1[n] + x_2[n] \quad , N - \text{per}$

$N = \text{cmmmc}\{N_1, N_2\}.$

Here the period is

$N = \text{cmmmc}\{4, 5\} = 20. \quad \boxed{N = 20}$

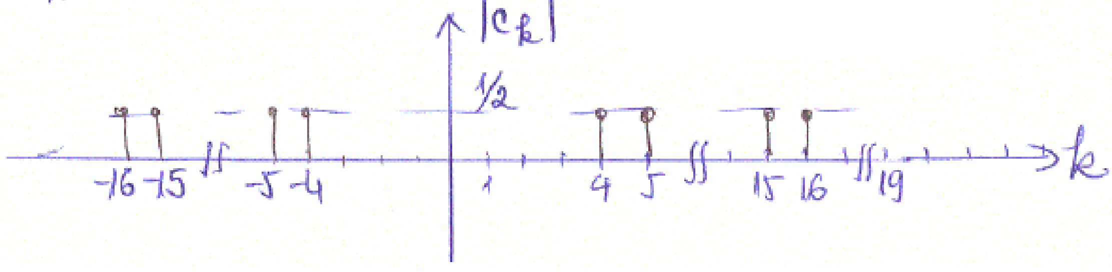
$\Rightarrow \quad x[n] = \sum_{k=0}^{19} c_k e^{jk\frac{\pi}{10} n}$

$$x[n] = \frac{1}{2} e^{j\frac{2\pi}{4}n} + \frac{1}{2} e^{-j\frac{2\pi}{4}n} + \frac{1}{2} e^{j\frac{2\pi}{5}n} + \frac{1}{2} e^{-j\frac{2\pi}{5}n}$$

$$= \underbrace{\frac{1}{2}}_{c_5} e^{j\cdot 5\frac{\pi}{10}n} + \underbrace{\frac{1}{2}}_{c_{20-5}=c_{15}} e^{-j5\frac{\pi}{10}n} + \underbrace{\frac{1}{2}}_{c_4} e^{j4\frac{\pi}{10}n} + \underbrace{\frac{1}{2}}_{c_{20-4}=c_{16}} e^{-j4\frac{\pi}{10}n}$$

$$\begin{cases} c_4 = c_5 = c_{15} = c_{16} = \frac{1}{2} & \arg c_k = 0. \\ c_k = 0 & \text{rest} \end{cases}$$

Magnitude spectrum



c) $x[n] = e^{-2n}$ $\quad N=4 \quad n \in \{-2,-1,0,1\}$

$$x[n] = \sum_{k=0}^{3} c_k e^{jk\frac{\pi}{2}n}$$

$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n] e^{-jk\frac{2\pi}{N}n} \qquad k = \overline{0, N-1}$$

--homework