3. Transversal digital filters

1. Application goal

We study the behavior of transversal digital filters in both time and frequency domain. A particular case of these filters is the moving average filter (running averager). We study this system from the point of view of a matched filter.

2. Transversal digital filters

The difference equation that describes a digital filter with input $s_i[n]$, and output $s_0[n]$ is:

$$\sum_{k=0}^{N} a_k s_i[n-k] = \sum_{k=0}^{M} b_k s_0[n-k] \tag{1}$$

We consider null initial conditions.

If the only coefficient a_k not null is a_0 then the filter is a non-recursive system. The length of their impulse response is finite, which is why they are called „finite impulse response" (FIR). For $a_0 = 1$, the difference equation of the system is:

$$s_0[n] = \sum_{k=0}^{M} b_k s_i[n-k] \tag{2}$$

The direct form I of implementation for the system in (2) is shown in Fig. 1.

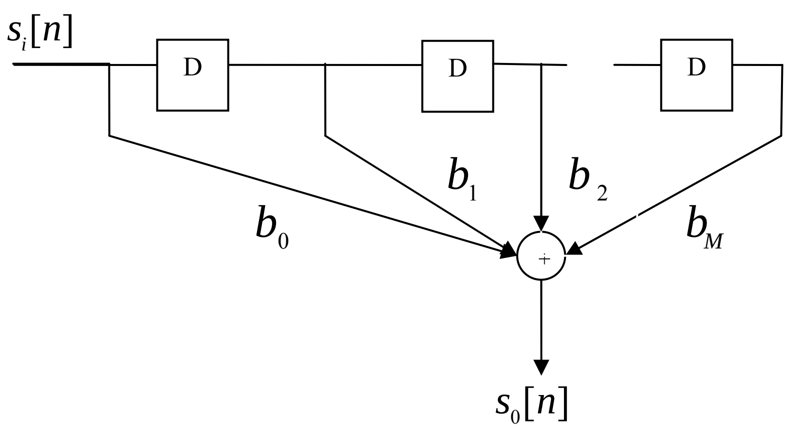

Fig.1 Direct form I of implementation for a FIR system.

The figure shows that the operations of delay, multiplication by a constant and summation are carried out on transverse directions; hence the name "transversal digital filter".

The impulse response of the transversal digital filter is:

$$h[n] = \sum_{k=0}^{M} b_k \delta[n-k] \tag{3}$$

The frequency response is:

$$H(\Omega) = \sum_{k=0}^{M} b_k e^{-jk\Omega} \tag{4}$$

3. Moving average filter (running averager)

If the coefficients $b_k = 1/M$ for $k = \overline{0, M-1}$ and are otherwise zero we have the difference equation:

$$s_0[n] = \frac{1}{M}\sum_{k=0}^{M-1} s_i[n-k] \tag{5}$$

The signal $s_0[n]$ is obtained from $s_i[n]$, by averaging the M samples of the input, including the current sample. This is a moving average filter or a running averager; its impulse response is:

$$h_0[n] = \frac{1}{M}\sum_{k=0}^{M-1} \delta[n-k] \tag{6}$$

Its frequency response is:

$$H(\Omega) = \frac{1}{M}\sum_{k=0}^{M-1} e^{-jk\Omega} = \frac{1}{M} e^{-j(M-1)\frac{\Omega}{2}} \frac{\sin\left(M\frac{\Omega}{2}\right)}{\sin\frac{\Omega}{2}} \tag{7}$$

For $M = 4$ the impulse response and the magnitude spectrum are shown in Fig. 2.

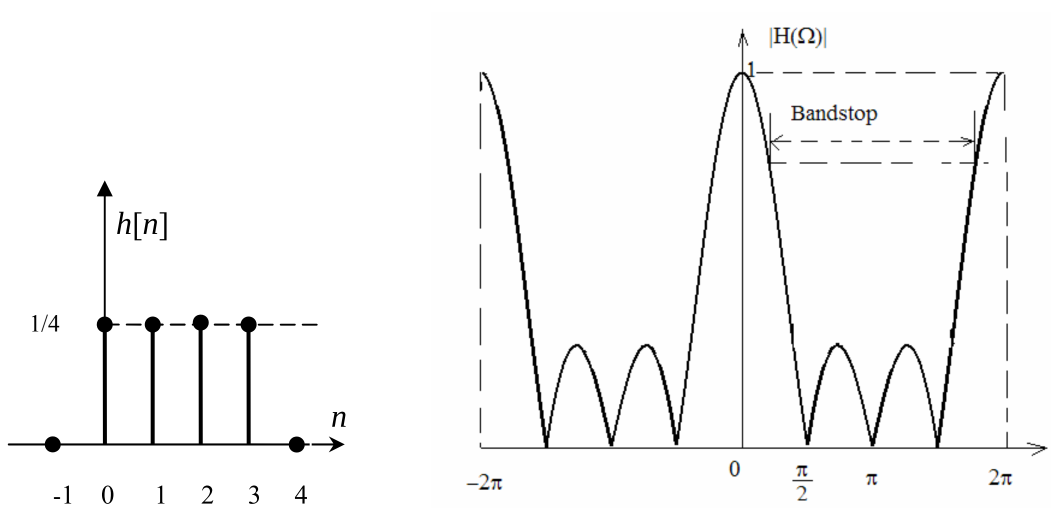

Fig.2 Impulse response and magnitude spectrum of the running averager, $M = 4$.

The running averager is a lowpass filter, with infinite attenuation at certain frequencies. The frequencies and the minimum attenuation in the bandstop depend on the value of M.

From eq. (5): each sample $s_0[n]$ is the average obtained by a sliding window on the last M samples of the input $s_i[n]$ (hence the name running averager).

4. Adaptive digital filters

Consider that we have to transmit the useful signal $s_i[n]$, of finite duration M, that is perturbed with noise $n_i[n]$:

$$x[n] = s_i[n] + n_i[n] \tag{8}$$

In order to remove the noise from the signal, we can process it using a nonrecursive discrete-time system $h[n]$, and we obtain the signal $y[n]$:

$$y[n]=s_0[n]+n_0[n] \tag{9}$$

where $s_0[n]$ is the useful component of the signal $x[n]$ and $n_0[n]$ is random component. The problem is finding the impulse response $h[n]$ of this system, that maximizes the output signal-to-noise ratio (SNR) at the moment of time n_0. The system that fulfills this condition is named matched filter for the signal $s_i[n]$. The output signal-to-noise ratio (SNR_0) should be time-dependant, in order to be maximized at the moment n_0. It is computed as:

$$SNR_0=\frac{s_0^2[n]}{P_{n0}[n]} \tag{10}$$

where $P_{n0}[n]$ is the power of the output signal. But:

$$s_0[n]=s_i[n]*h[n]=\sum_{p=0}^{M-1}s_i[p]\cdot h[n-p] \tag{11}$$

Using Cauchy-Buniakovsky-Schwarz inequality, we have:

$$s_0^2[n]\le\left(\sum_{p=0}^{M-1}s_i^2[p]\right)\left(\sum_{p=0}^{M-1}h^2[n_0-p]\right) \tag{12}$$

Equality takes place if:

$$h[n_0-p]=as_i[p] \quad \forall p=\overline{0,M-1} \tag{13}$$

where a is a constant. The system for which the input is $s_i[n]$ and the output is $s_0[n]$, maximized at moment n_0, has the impulse response:

$$h[n]=as_i[n_0-n] \qquad n=\overline{n_0-(M-1),n_0} \tag{14}$$

If $n_0=M-1$, then:

$$h[n]=as_i[M-1-n] \qquad n=\overline{0,M-1} \tag{15}$$

We consider that $n_0[n]$ is a random signal, of type white noise whose samples are not correlated with the samples of the signal $s_i[n]$ and are independent. The nonrecursive system with the impulse response of duration M that maximizes the output signal-to-noise ratio SNR_0 is the one given in eq. (14). Hence this is a matched filter to the signal $s_i[n]$ in the hypothesis that the signal is perturbed with additive white noise.

If the useful component at the input of the filter is:

$$s_i[n]=\begin{cases}1, n=\overline{0,M-1}\\0, \text{otherwise}\end{cases} \tag{16}$$

and if the constant $a=1/M$, then:

$$h[n]=\begin{cases}1/M, n=\overline{0,M-1}\\0 \quad ,\text{otherwise}\end{cases} \tag{17}$$

The running averager is matched filter for the signal in (16). It is of finite duration and amplitude constant. It is useful for increasing the signal-to-noise ratio of periodic continuous-time signals.

5. Practical part

The purpose of this lab is the simulation in Matlab of the system in Fig. 1. The useful signal at the input of the system is a discrete time sine wave, with *N* samples per period (maximum 512). The number M, power of 2, has values of 2, 4, 8, 16, 32, 64.

5.1. Using the program *"ftrans"*, visualize and plot the impulse response of the system, $h[n]$ and the input and output signals $s_i[n]$ and $s_0[n]$ for M = 2, 4, 8, 16, 32 and 64.

5.2. Determine the frequency characteristics of the system simulated at 5.1. and plot it on graph paper. Compare with the result from Matlab.

5.3. Add white noise with a known power, to the input signal $s_i[n]$ and obtain the signal $x[n]$. The input signal-to-noise ratio SNRi is computed as the ratio between the squared RMS value of the signal $s_i[n]$ and the power of the noise.

Choose the length of the filter M=2, 4, 8, 16, 32, 64. Compute the output signal-to-noise ratio SNRo, ratio between the squared RMS value of the signal $s_0[n]$ and the power of the noise.

Compute the signal-to-noise ratio enhancement Γ as the ratio SNRo/SNRi, for each value of M. Plot on graph paper the dependency $\Gamma(M)$.

6. Exercises in MATLAB

6.1. Gibbs phenomenon

6.1.1. The square wave is expanded into a series of sine waves. Consider the sum of the periodic signals: $x(t)=\frac{4}{\pi}\left[\sin\omega_0 t+\frac{1}{3}\sin 3\omega_0 t+\frac{1}{5}\sin 5\omega_0 t+\ldots\right]=\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin n\omega_0 t$, where *n* is an odd number.

The following Matlab commands use a loop to generate this sum for any odd number:

```
n=input('Enter an odd number:    ');
omega_0=0:0.01:2*pi;
x=0;
for k=1:2:n;
x=x+1/k*sin(k*omega_0);
end
x=4/pi*x;
plot(omega_0,x), xlabel('omega_0')
text(3.5,.7,['Sum of ', num2str((n+1)/2), ' sine waves '])
```

Run the sequence for different values of odd numbers.

6.2. Discrete-time Fourier transform

6.2.1. Compute the discrete-time Fourier transform of the sequence: $x_1[n]=\sigma[n]-\sigma[n-8]$, for $0\le n\le 20$. Plot the real part, the imaginary part, the absolute value and the phase of the computed DTFT.

```
n=0:20;
x1=[ones(1,8), zeros(1,13)];
stem(x1)
X=fft(x1,512); %see help fft
```

The DTFT computation was made on 512 points.

```
plot(abs(X)),grid; % see help abs
```

This representation corresponds to the interval of frequencies [0,2π). Using the command `fftshift` we are able to plot the DTFT in the interval [-π, π). We generate a vector with linear step. Its length is the length of the 512-point discrete time Fourier transform.

```
w=-pi:2*pi/512:pi-2*pi/512;
plot(w,fftshift(abs(X))),grid %see help fftshift

plot(w,fftshift(angle(X))),grid

subplot(2,1,1), plot(w,fftshift(real(X))),grid
subplot(2,1,1), plot(w,fftshift(imag(X))),grid
```

Exercises

Using the example above, find and plot the real part, the imaginary part, the magnitude and the phase of the discrete-time Fourier transform for the signals:

$x_1[n]=\delta[n]$ for $0\le n\le 10$

$x_2[n]=\delta[n-1]-\delta[n-3]+\delta[n-5]$ for $0\le n\le 10$

$x_3[n]=\sin\left(\frac{n\pi}{5}\right)$ for $0\le n\le 20$

$x_4[n]=\cos\left(\frac{n\pi}{5}\right)$ for $0\le n\le 20$

6.3. The impulse response of a linear-time invariant system in discrete-time

6.3.1. Find and plot the impulse response of a discrete-time linear-time invariant (LTI) system defined by:

$$y[n]-0.9y[n-1]=0.3x[n]+0.6x[n-1]+0.6x[n-2]$$

Solution

```
b=[0.3,0.6,0.6];
a=[1,-0.9];
```

```
[h,t]=impz(b,a) %the impulse response of the system is computed.
                %See help for impz
impz(b,a), grid
stem(t,h), grid
```

Verify and explain the effect of the command:

```
impz(b,a,40,3), grid
```

Exercises

Find and plot the impulse response of a system defined by:

1. $y[n]=x[n]-1.27x[n-1]+0.81x[n-2]-0.5x[n-3]+0.125x[n-4]$
2. $y[n]+0.9y[n-1]=x[n]$
3. $y[n]+0.13y[n-1]+0.52y[n-2]+0.3y[n-3]=0.16x[n]-0.48x[n-1]+0.48x[n-2]-0.16x[n-3]$
4. $y[n]+0.9y[n-2]=0.3x[n]+0.6x[n-1]+0.3x[n-2]$

6.4. The response of a linear-time invariant system in discrete-time

1. Find and plot the response of the system defined by the equation:

$$y[n]-0.9y[n-1]=0.3x[n]+0.6x[n-1]+0.6x[n-2]$$

at the input signal $x_1[n]=\sigma[n]-\sigma[n-10]$ for $0\le n\le 40$.

Solution

```
b=[0.3,0.6,0.6]
a=[1,-0.9]
x=[ones(1,10), zeros(1,31)];
y=filter(b,a,x); %see help filter
n=0:40;
subplot(3,1,1), stem(n,x), grid, title('x[n]')
subplot(3,1,2), impz(b,a), grid, title('h[n]')
subplot(3,1,3), stem(n,y), grid, title('y[n]')
```

Exercises

Find the response of the systems defined below

1. $y[n]=x[n]-1.27x[n-1]+0.81x[n-2]-0.5x[n-3]+0.125x[n-4]$
2. $y[n]+0.9y[n-1]=x[n]$
3. $y[n]+0.13y[n-1]+0.52y[n-2]+0.3y[n-3]=0.16x[n]-0.48x[n-1]+0.48x[n-2]-0.16x[n-3]$
4. $y[n]+0.9y[n-2]=0.3x[n]+0.6x[n-1]+0.3x[n-2]$

to the following input signals:

1. $x_1[n]=\delta[n]$ for $0\le n\le 40$
2. $x_2[n]=\sigma[n]$ for $0\le n\le 40$
3. $x_3[n]=n$ for $0\le n\le 5$