

## 2. First and second order systems

### 1. Application goal

We study the behavior of first and second order systems in both time and frequency domain.

### 2. First order systems: Behavior in time

A first order system with a lowpass (LP) characteristic is described by the differential equation:

$$\tau \frac{dy(t)}{dt} + y(t) = x(t) \quad (1)$$

Here  $x(t)$  is the input signal,  $y(t)$  is the output signal.  $\tau$  is the time constant of the system. For null initial conditions (for zero input, the output is zero) eq. (1) describes a linear time-invariant system whose system function is:

$$H_{LP}(s) = \frac{1}{1 + s\tau}, \quad \text{Re}\{s\} > -\frac{1}{\tau} \quad (2)$$

We have:

- The impulse response (response at the unit impulse):

$$h(t) = \frac{1}{\tau} e^{-t/\tau} \sigma(t) \quad (3)$$

- The step response (response at the unit step signal):

$$s(t) = \left[ \int_0^t h(\theta) d\theta \right] \sigma(t) = (1 - e^{-t/\tau}) \sigma(t) \quad (4)$$

At  $t = \tau$ , the impulse response is at  $1/e$  of its value at  $t = 0$ .

A first order system with a highpass (HP) characteristic is described by the differential equation:

$$\tau \frac{dy(t)}{dt} + y(t) = \tau \frac{dx(t)}{dt} \quad (5)$$

For null initial conditions, the system function is:

$$H_{HP}(s) = \frac{s\tau}{1 + s\tau}, \quad \text{Re}\{s\} > -\frac{1}{\tau} \quad (6)$$

### 3. First order systems: Behavior in frequency

The frequency response of the first order system with a LP characteristic is:

$$H_{LP}(\omega) = H_{LP}(s) |_{s=j\omega} = \frac{1}{1 + j\omega\tau} \quad (7)$$

and of a first order system with a HP characteristic is:

$$H_{HP}(\omega) = H_{HP}(s) |_{s=j\omega} = \frac{j\omega\tau}{1 + j\omega\tau} \quad (8)$$

### 4. Second order systems with lowpass characteristic: Behavior in time

The differential equation that describes the second order systems with lowpass characteristic is:

$$\frac{d^2 y(t)}{dt^2} + 2\xi\omega_n \frac{dy(t)}{dt} + \omega_n^2 y(t) = \omega_n^2 x(t) \quad (9)$$

Here  $x(t)$  is the input signal,  $y(t)$  is the output signal.  $\xi$  is the damping ratio and  $\omega_n$  is the undamped natural frequency. For null initial conditions, eq. (9) describes a linear time-invariant system, whose system function is:

$$H(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} \quad \text{Re}\{s\} > \sigma_0 \quad (10)$$

The poles of system function are the roots of the equation below:

$$s^2 + 2\xi\omega_n s + \omega_n^2 = 0 \quad (11)$$

The poles are:

$$c_1 = -\xi\omega_n + \omega_n\sqrt{\xi^2 - 1} \quad (12)$$

$$c_2 = -\xi\omega_n - \omega_n\sqrt{\xi^2 - 1} \quad (13)$$

- **Under-damped system:** for  $0 < \xi < 1$ , the poles are different poles and complex conjugate pairs;  $\sigma_0 = -\xi\omega_n$ .
- **Critically damped system:** for  $\xi = 1$ , the poles are real and equal;  $\sigma_0 = -\omega_n$ .
- **Overdamped system:** for  $\xi > 1$ , the poles are real but different;  $\sigma_0 = -\xi\omega_n + \omega_n\sqrt{\xi^2 - 1}$ .

In the last two cases, the second order system is equivalent to a series interconnection of two first order systems. The impulse response is obtained using the inverse Laplace transform in eq. (10). If  $\xi \neq 1$  we have:

$$h(t) = M \left[ e^{c_1 t} - e^{c_2 t} \right] \sigma(t) \quad (14)$$

where:

$$M = \frac{\omega_n}{2\sqrt{\xi^2 - 1}} \quad (15)$$

For  $\xi = 1$  we obtain:

$$h(t) = \omega_n^2 t e^{-\omega_n t} \sigma(t) \quad (16)$$

For  $0 < \xi < 1$  relation (14) becomes:

$$h(t) = \frac{A\omega_n}{\sqrt{1-\xi^2}} e^{-\xi\omega_n t} \sin(\omega_n\sqrt{1-\xi^2}t) \sigma(t) \quad (17)$$

Function  $h(t)$  is represented in Fig. 1 for different values of the damping ratio  $\xi$ .

The step response is the response of the system to the unit step  $\sigma(t)$  and it is denoted by  $s(t)$ . If  $\xi \neq 1$ , the step response is:

$$s(t) = h(t) * \sigma(t) = \int_{-\infty}^t h(\tau) d\tau = \left[ 1 + M \left( \frac{e^{c_1 t}}{c_1} - \frac{e^{c_2 t}}{c_2} \right) \right] \sigma(t) \quad (18)$$

If  $\xi = 1$ :

$$s(t) = (1 - e^{-\omega_n t} - t e^{-\omega_n t}) \sigma(t) \quad (19)$$

If  $0 < \xi < 1$ :

$$s(t) = A \left[ 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin(\omega_n\sqrt{1-\xi^2}t + \arccos \xi) \right] \sigma(t) \quad (20)$$

Function  $s(t)$  is represented in Fig. 2 for different values of the damping ratio  $\xi$ .

We see that in the underdamped case, the step response exceeds its final value and has an oscillatory behavior. In the overdamped case, its time of response becomes longer (slow response).

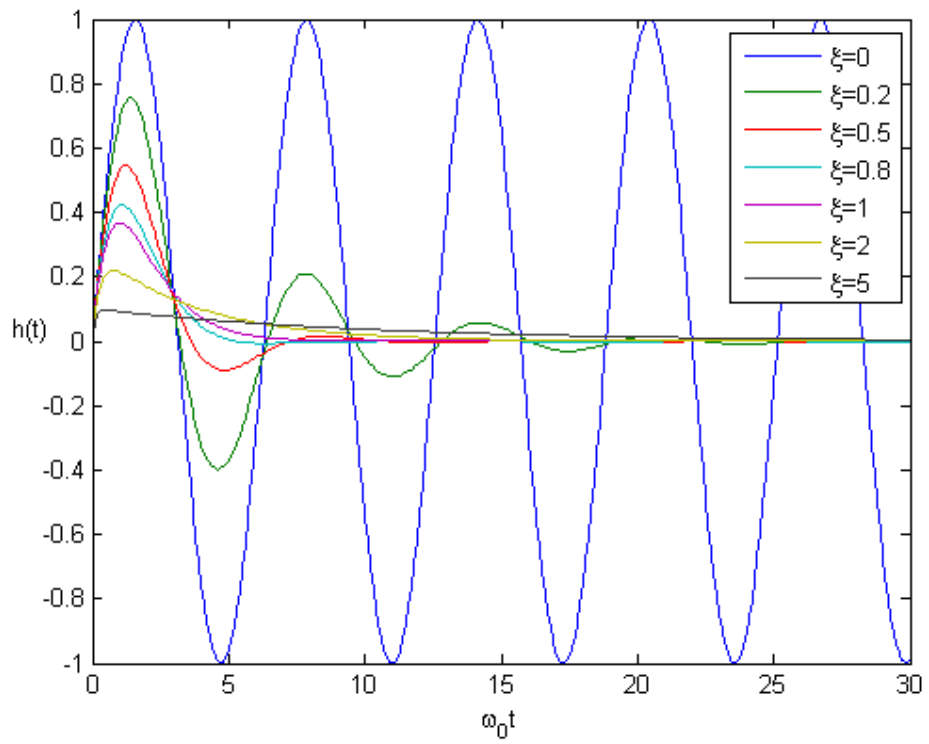


Fig. 1 Impulse response for different values of the damping ratio  $\xi$  (second order system).

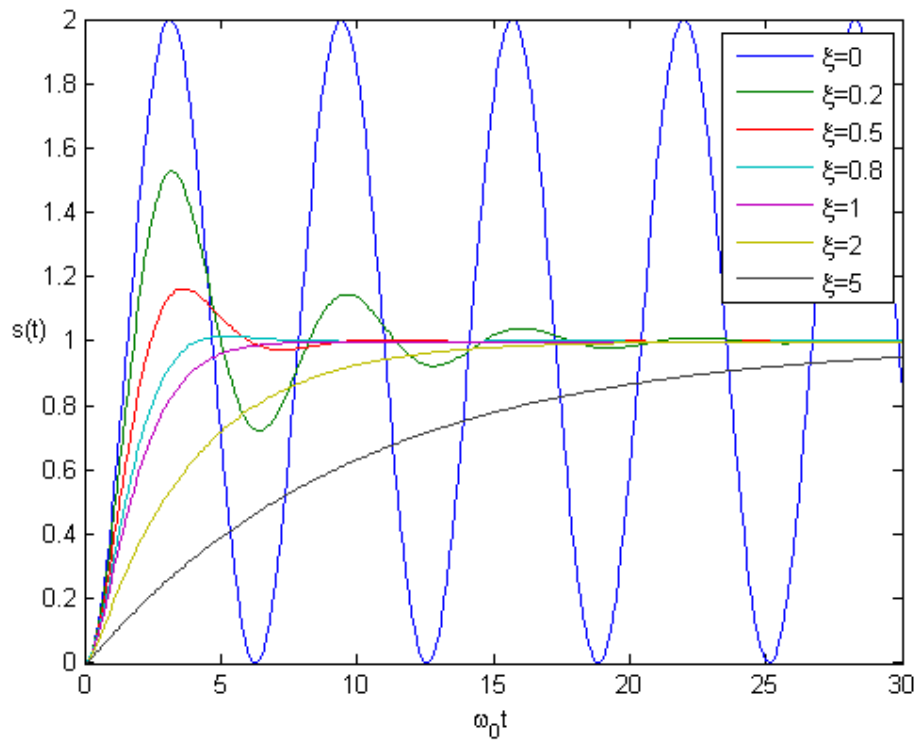


Fig. 2 Step response for different values of the damping ratio  $\xi$  (second order system).

## 5. Second order systems with lowpass characteristic: Behavior in frequency

The frequency response of a second order system with lowpass characteristic is obtained from (10) for  $s = j\omega$ :

$$H(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\xi\omega_n j\omega + \omega_n^2} \quad (21)$$

Bode plots are shown in Fig. 3. In the underdamped case the magnitude characteristic has a maximum value at:

$$\omega_{\max} = \omega_n \sqrt{1 - 2\xi^2} \quad (22)$$

if  $\xi < \sqrt{2}/2$ .

For  $\xi \ll 1$  we can consider:

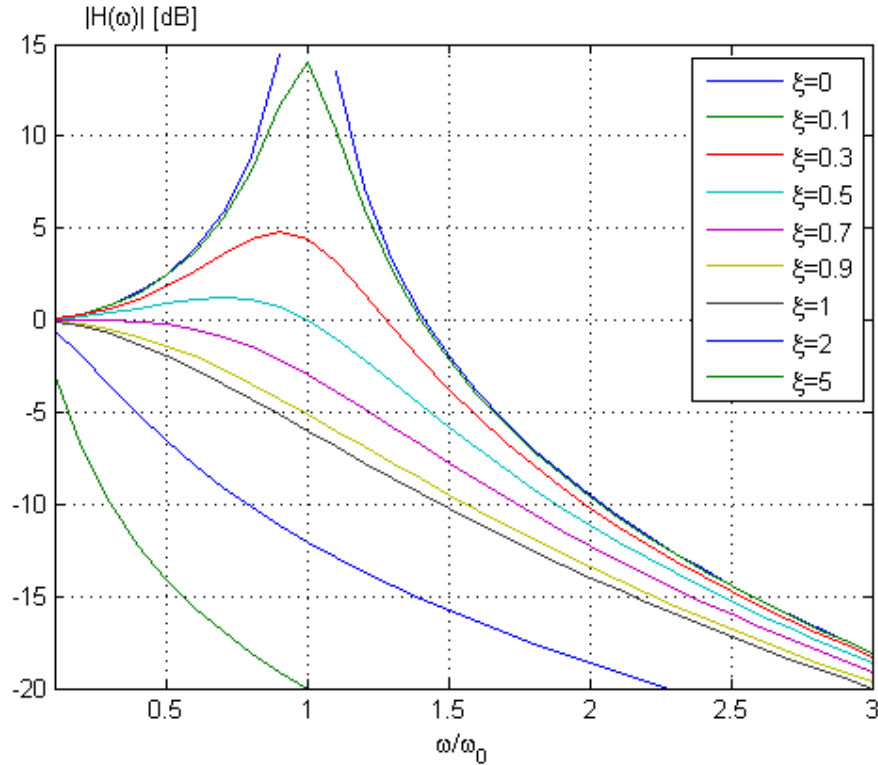
$$\omega_{\max} \cong \omega_n \quad (23)$$

The maximum is given by:

$$|H(\omega_{\max})| = 1/(2\xi\sqrt{1 - \xi^2}) \quad (24)$$

For  $\xi \ll 1$  we obtain:

$$|H(\omega_{\max})| \cong \frac{1}{2\xi} \quad (25)$$



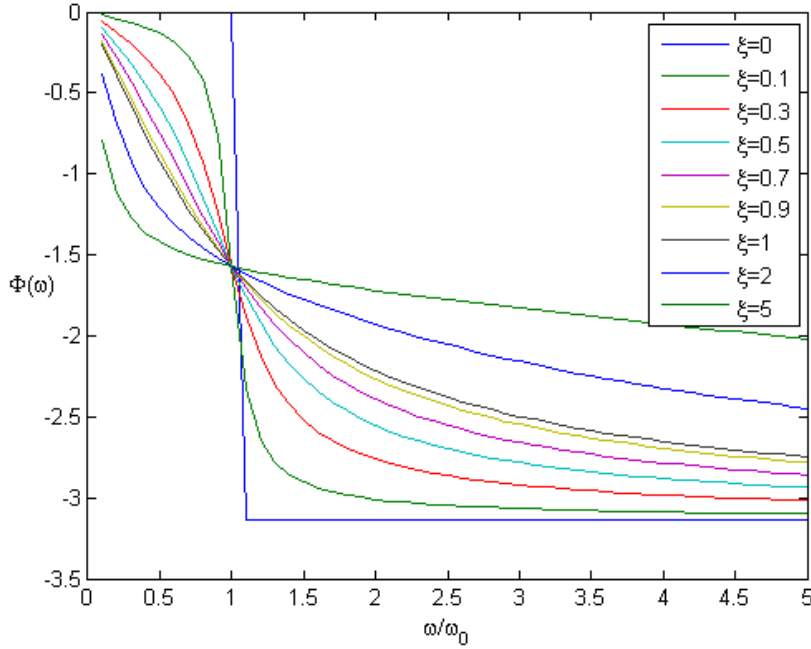


Fig. 3 Bode plots for different values of the damping ratio  $\xi$  (second order LP system).

If

$$20 \log |H(\omega_{\max})| \geq 3 \text{ dB} \quad (26)$$

then the system behaves like a bandpass filter. This happens for:

$$0 < \xi \leq \sqrt{\frac{4 - \sqrt{8}}{8}} = 0.38 \quad (27)$$

In this case, the bandwidth at 3 dB can be approximated with:

$$B = 2\xi\omega_n \quad (28)$$

We define the quality factor of a second order system with a lowpass characteristic:

$$Q = \frac{1}{2\xi} \quad (29)$$

## 6. Types of second order systems

- lowpass

$$H_{LP}(s) = \frac{A_{TL}\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} \quad \text{Re}\{s\} > \sigma_0 \quad (30)$$

- highpass

$$H_{HP}(s) = \frac{A_{TS}s^2}{s^2 + 2\xi\omega_n s + \omega_n^2} \quad \text{Re}\{s\} > \sigma_0 \quad (31)$$

- bandpass

$$H_{TB}(s) = \frac{A_{TB}s \cdot 2\xi\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} \quad \text{Re}\{s\} > \sigma_0 \quad (32)$$

- bandstop

$$H_{BS}(s) = \frac{A_{OB}(s^2 + \omega_n^2)}{s^2 + 2\xi\omega_n s + \omega_n^2} \quad \text{Re}\{s\} > \sigma_0 \quad (33)$$

For values of  $\sigma_0$  see comments after eq. (13).

## 7. Practical part

We use

- signal generator,
- oscilloscope
- filter: lowpass filter LPF – green led; bandpass filter BPF – yellow led; highpass filter HPF – red led
- power source ( $\pm 12V$ ) for the filter.

The signal generator's output  $50\Omega$  is connected to the input of the oscilloscope (channel 1) and to the input of the filter. The output of the filter is visualized on channel 2 of the oscilloscope.

**7.1** Determine experimentally the frequency characteristics (magnitude and phase spectra) for three types of second order systems: lowpass, highpass and bandpass. Initial data:

$U_{in} = 1V$ , the amplitude of the input signal (that means peak-to-peak 2V)

$f_c = 3.4 \text{ kHz}$ , cutoff frequency of the filter

Fill in the table for the three filters:

| $f [kHz]$       | 0.1 | 0.4 | 1.4 | 2.4 | 2.9 | 3.2 | 3.4 | 3.6 | 3.9 | 4.4 | 5.4 | 6.4 | 7.9 | 10 | 20 |
|-----------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|----|----|
| $U_{out} [V]$   |     |     |     |     |     |     |     |     |     |     |     |     |     |    |    |
| $\Delta T [s]$  |     |     |     |     |     |     |     |     |     |     |     |     |     |    |    |
| $T [s]$         |     |     |     |     |     |     |     |     |     |     |     |     |     |    |    |
| $\varphi [rad]$ |     |     |     |     |     |     |     |     |     |     |     |     |     |    |    |

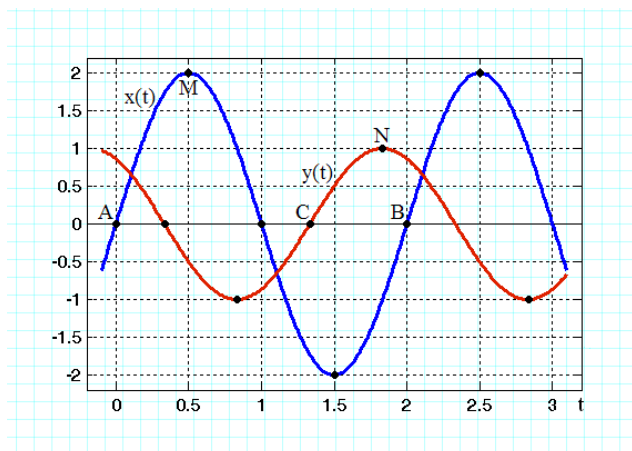
Please keep in mind that the phase is computed using the rule of three:

$$\Delta T [s] \dots\dots\dots \varphi [rad]$$

$$T [s] \dots\dots\dots -2\pi$$

$$\varphi [rad] = \frac{-2\pi\Delta T}{T}$$

$\Delta T [s]$  and  $T [s]$  are read using the cursor on the oscilloscope.



Example of measurement:  $\Delta T [s]$  is the segment AC and the period  $T [s]$  is the segment AB.  $x(t)$  is the input signal while  $y(t)$  is the output signal.

7.2 Represent on graph paper magnitude and phase spectra:

$$\frac{U_{out}[V]}{U_{in}[V]} = function(kf)$$

$$\phi[rad] = function(kf)$$

Total: 6 graphs.

## 8. Exercises in Matlab

### Sine wave

Plot the functions  $f(t) = \sin(2\pi 50t)$  and  $g(t) = -f(t)$ , using the title „Graphical representation of functions  $f(t)$  and  $g(t)$ ”; on the  $x$ -axis „ $t$ ”, and on the  $y$ -axis write  $f(t)$  and  $g(t)$ .

```
t=0:.001:0.02
f=sin(2*pi*50*t)
g=-f
plot(t,f,t,g,'g'),grid on
title(' Graphical representation of functions f(t) and g(t) ')
xlabel('t'), ylabel('f(t) and g(t)')
```

### Exercise

Plot the discrete sequence:  $x[n] = \sin\left(2\pi \frac{1}{10}n\right)$ , for  $n \in [0, 20]$  in red color. Write the title and axes. Use *stem* command.

### Convolution of signals

Compute the convolution between two sequences:  $x[n] = \delta[n] + \delta[n-1] + \delta[n-2]$  and  $h[n] = \delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3]$ . Convolution is defined as:  $y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$ .

```
x=[1 1 1]; h=[1 1 1 1]
disp('Result of convolution:')
y=conv(x,h)
stem(y)
```

Compute and plot the convolution between the following two sequences:

$$x[n] = \sigma[n] - \sigma[n-5], \text{ for } 0 \leq n \leq 10$$

$$h[n] = (0,9)^n, \text{ for } 0 \leq n \leq 20$$

```
x=[ones(1,5), zeros(1,6)];
n=0:20;
h=0.9.^n;
y=conv(x,h);
subplot(2,2,1),stem(0:10,x),title('x'),grid
subplot(2,2,2),stem(n,h),title('h'),grid
subplot(2,1,2),stem(0:length(y)-1,y),title('y'),grid
```

### Exercises

1. We have the sequences:

$$x[n] = \begin{cases} 1, & \text{if } n = 0, 1, 2, 3, 4, 5 \\ 0, & \text{otherwise} \end{cases}$$

$$h[n] = \begin{cases} n+1, & \text{if } n = 0, 1, 2 \\ 0, & \text{otherwise} \end{cases}$$

Compute  $y[n] = x[n] * h[n]$ , using the function *conv*, and plot the result.

2. We have the sequences:

$$x[n] = \begin{cases} 1, & \text{if } n = 0, 1, 2 \\ 0, & \text{otherwise} \end{cases}$$

$$h[n] = \begin{cases} 5-n, & \text{if } n = 0, 1, 2, 3, 4 \\ 0, & \text{otherwise} \end{cases}$$

Compute  $y[n] = x[n] * h[n]$ , using the function *conv*, and plot the result.

### Periodic signals

MATLAB cannot generate sequences of infinite length; the number of periods must be limited.

1.  $x_1[n] = n$  for  $0 \leq n \leq 5$  (3 periods)

```
n=0:5;
x1=n;
figure(1)
stem(n,x1),grid
x11=[x1,x1,x1];
figure(2)
stem(0:(length(x11)-1),x11),grid
```

### Exercises

Define and plot the following sequences:

1.  $x_2[n] = \sigma[n-2] - \sigma[n-4]$  for  $0 \leq n \leq 5$  (5 periods)

2.  $x_3[n] = \delta[n-1] - \delta[n-3]$  for  $0 \leq n \leq 5$  (7 periods)

### Complex signals

1. We define the discrete-time complex signal:

$$x_1[n] = e^{j\frac{\pi}{5}n} \text{ for } -20 \leq n \leq 20$$

Plot the even and odd part of this sequence.

```
n=-20:20;
x1=exp(j*n*pi/5);
subplot(2,1,1),stem(n,real(x1)),title('Real')
subplot(2,1,2),stem(n,imag(x1)),title('Imaginary')
```

### Exercise

We define the discrete-time complex signal:

$$x_2[n] = 3n - j(n-1)^2 \text{ for } -20 \leq n \leq 20$$

Plot the even and odd part of this sequence.

Explain the result obtained after the following lines:

```
plot(x1)
plot(x2)
```