1. Periodic signals

1. Application goal

Periodic signals (square, sine, triangle wave) are analyzed in the frequency domain. Signal distortion is measured at the output of signal generators for a sine wave.

2. Exponential Fourier expansion of periodic complex signals

A continuous-time signal $x : \mathrm{R} \to \mathrm{C}$ is periodic if there is a constant strictly positive T such that:

$$x(t+T) = x(t), \quad \forall t \in \mathrm{R} \tag{1}$$

The period is the smallest value of T that fulfills the equality from relation (1).

Property: If a function $x(t)$ is locally integrable and periodic of period T, then:

$$\int_0^T x(t)\,dt = \int_a^{a+T} x(t)\,dt, \quad \forall a \in \mathrm{R} \tag{2}$$

The integral computed over one period of the function does not depend on the starting point of the interval of integration. This is simply denoted with:

$$\int_T x(t)dt$$

A periodic signal of period T, satisfies Dirichlet conditions if:

- $x(t)$ is absolutely integrable over one period:

$$\int_T |x(t)|dt < \infty$$

- $x(t)$ have a finite number of extrema (maxima and minima) in any given interval T.
- $x(t)$ has a finite number of first-order discontinuities in any given interval T.
- $x(t)$ is bounded.

If a periodic signal of period T, fulfills Dirichlet conditions, then it can be written as an exponential Fourier expansion:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k \cdot e^{jk\omega_0 t}, \quad \omega_0 = \frac{2\pi}{T} \tag{3}$$

with

$$c_k = \frac{1}{T}\int_T x(t) e^{-jk\omega_0 t} dt \tag{4}$$

where c_k are the exponential Fourier coefficients.

3. Trigonometric and harmonic Fourier expansion for periodic real signals

Consider periodic real signals $x : \mathrm{R} \to \mathrm{R}$, of period T, that satisfy Dirichlet conditions then equation (3) can be written as:

- trigonometric Fourier series

$$x(t) = a_0 + \sum_{k=1}^{\infty}\left(a_k \cdot \cos k\omega_0 t + b_k \cdot \sin k\omega_0 t\right) \tag{5}$$

- harmonic Fourier series

$$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cdot \cos(k\omega_0 t + \varphi_k) \tag{6}$$

Relations between coefficients from (3), (5) and (6) are:

$$\begin{aligned} c_0 &= a_0 = A_0 \\ a_k &= A_k \cdot \cos\varphi_k = 2|c_k| \cdot \cos(\arg\{c_k\}), \quad k > 0 \\ b_k &= -A_k \cdot \sin\varphi_k = -2|c_k| \cdot \sin(\arg\{c_k\}), \quad k > 0 \end{aligned} \tag{7}$$

About relation (6):

- A_0 is the ***DC offset component*** of the signal $x(t)$
- The signal $A_k \cdot \cos(k\omega_0 t + \varphi_k)$ is the k^{th} ***order harmonic*** of signal $x(t)$, $k > 0$
- $\frac{A_k}{\sqrt{2}}$ are root mean square (RMS) values of harmonics ($k > 0$)
- The ***amplitude spectrum*** of a periodic signal $x(t)$ is the graphical representation of A_k as a function of $k\omega_0$, $k \geq 0$
- The ***phase spectrum*** of a periodic signal $x(t)$ is the graphical representation of φ_k as a function of $k\omega_0$, $k \geq 0$.

4. Power of periodic signals

Consider a signal $x(t)$. If the following limit converges:

$$P_X = \lim_{\tau \to \infty} \frac{1}{2\tau} \int_{-\tau}^{\tau} |x(t)|^2 dt$$

it is called the power of the signal $x(t)$.

We can show that if $x(t)$ is a periodic signal of period T, then we its power is computed over one period:

$$P_X = \frac{1}{T} \int_T |x(t)|^2 dt \tag{8}$$

The following Parseval relation is valid:

$$P_X = \sum_{k=-\infty}^{\infty} |c_k|^2 = A_0^{\ 2} + \sum_{k=1}^{\infty} \left(\frac{A_k}{\sqrt{2}}\right)^2 \tag{9}$$

The first equality holds for any periodic signal; while the second one holds only for real signals. Because the power of a sinusoid is the square of its RMS value then the power of the periodic real signal is the sum of the power of the DC offset and the power for each sinusoid (harmonics) in the harmonic Fourier expansion.

5. Relation between Fourier coefficients of a periodic signal and the Fourier transform of the corresponding aperiodic signal

Consider $x(t)$ a periodic signal, of period T and $x_1(t)$ the signal:

$$x_1(t) = \begin{cases} x(t), & t \in \left[-\frac{T}{2}; \frac{T}{2}\right] \\ 0, & \text{otherwise} \end{cases} \tag{10}$$

Consider $X_1(\omega)$ the Fourier transform of the signal $x_1(t)$ and c_k the coefficients of the exponential Fourier expansion of the signal $x(t)$. We have:

$$c_k = \frac{1}{T} X_1(k\omega_0), \quad \omega_0 = \frac{2\pi}{T}, \ k \in \mathbb{Z} \tag{11}$$

6. Example

Consider the square wave in figure 1.

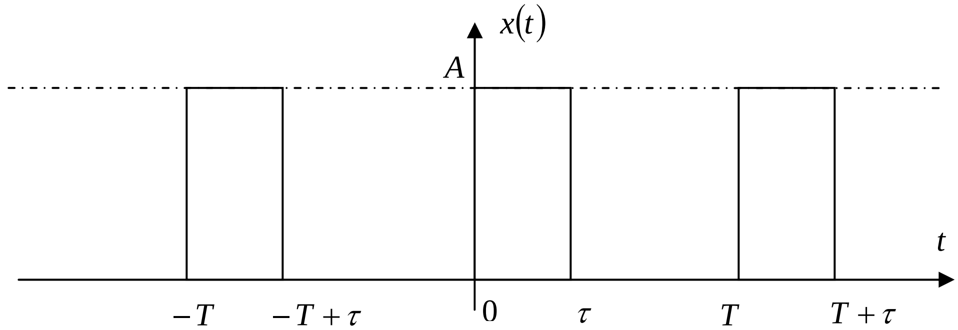

Fig.1 Square wave

The coefficients of the exponential Fourier series are in this case:

$$c_k = \frac{1}{T}\int_0^{\tau} A \cdot e^{-jk\omega_0 t} dt = A\frac{\sin k\omega_0 \tau / 2}{k\pi} \cdot e^{-jk\omega_0 \tau/2} \tag{12}$$

The power of the square wave is:

$$P_X = \frac{1}{T} A^2 \cdot \tau \tag{12`}$$

Taking into account (7) we have:

$$\begin{aligned} A_0 &= c_0 = A\frac{\tau}{T} \\ A_k &= 2|c_k| = 2A\left|\frac{\sin k\omega_0 \tau / 2}{k\pi}\right| \\ \varphi_k &= \arg\{c_k\} = -\frac{k\omega_0\tau}{2} + \left[1 - \operatorname{sgn}\left(\sin\frac{k\omega_0\tau}{2}\right)\right] \cdot \frac{\pi}{2}, \quad k > 0 \end{aligned} \tag{13}$$

Amplitude and phase spectra are shown in figure 2 for the case of $\tau / T = 1/4$ (duty cycle).

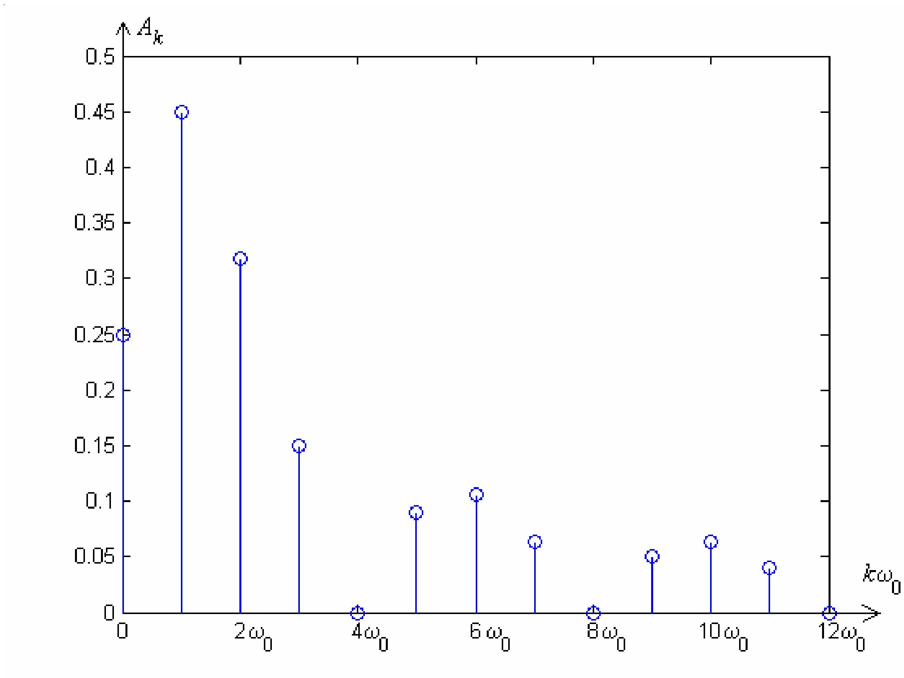

Fig 2a) Amplitude spectrum of the square wave, $\tau / T = 1/4$.

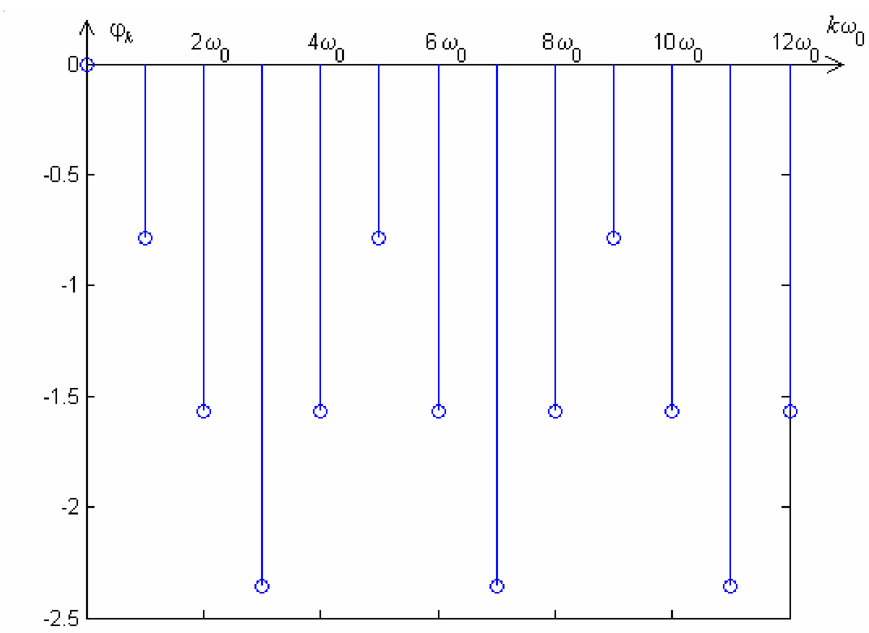

Fig 2b) Phase spectrum of the square wave, $\tau / T = 1/4$.

7. Measurement of the signal distortion

If a sinusoidal signal is brought at the input of an analog linear time-invariant system, the output is also sinusoidal.
In practice, the output is approximately sinusoidal, due to nonlinearities of real circuits. Consider the input signal:

$$x(t) = A \cdot \cos(\omega_0 t + \psi) \tag{14}$$

and the output signal

$$y(t) = A_0 + \sum_{k=1}^{\infty} A_k \cdot \cos(k\omega_0 t + \varphi_k) \tag{15}$$

The distortion factor is defined as:

$$d_Y = \frac{\sqrt{\sum_{k=2}^{\infty} \left(\frac{A_k}{\sqrt{2}} \right)^2}}{A_1 / \sqrt{2}} = \frac{\sqrt{\sum_{k=2}^{\infty} P_k}}{\sqrt{P_1}} \tag{16}$$

It is square root of the ratio sum of the powers of all harmonic frequencies above the fundamental frequency, to the power of the fundamental harmonic.

8. Practical part

We use a signal generator, an oscilloscope and a spectrum analyzer. The signal generator is the source of a periodic signal whose waveform can be sine wave, square wave or triangle wave. The signal generator's output 50Ω is connected to the input of oscilloscope (channel 1) and to the input of the spectrum analyzer.

8.1. Visualize and sketch on graph paper the waveforms for periodic signals generated using a function generator (square, sinusoidal, triangle wave). The signals are visualized in time on the oscilloscope and in frequency using the spectral analyzer.

Signal generator:

- fundamental frequency f_0=1MHz
- peak-to-peak amplitude A_{pp}=1V
- square wave: duty factor 0.5

Oscilloscope:

- Amplitude scale 0.5V/div; Time scale 0.5μs/div

Spectral analyzer:

Center frequency: 5MHz;	Reference level: 10dBm
Span width: 1MHz/div;	RBW: 200kHz

8.2 Compute Fourier coefficients A_k[V] for the harmonic series for the
sinusoidal wave $x(t) = 0.5V \cdot \sin \omega_0 t$ (use identification),
square wave with duty factor 0.5 (use the example in figure 1),
triangle wave.

Obs. The triangle wave can be written as:

$$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cdot \cos(k\omega_0 t + \varphi_k) = \frac{8}{\pi^2}\left(\sin\omega_0 t - \frac{1}{3^2}\sin 3\omega_0 t + \frac{1}{5^2}\sin 5\omega_0 t - \frac{1}{7^2}\sin 7\omega_0 t + ...\right)$$

8.3 Measure the RMS values of the harmonics for all signals using a spectrum analyzer. Use **"Marker"** and **"Shift Marker"** to read the values in dBm. Do not forget the values are either positive or negative.
Fill in the tables:

Table I. Sine wave

F	U_{RMS} [dBm]	U_{RMS}[mV]	U[mV]	A_k[mV]
f_0				
$2f_0$				
$3f_0$				
$4f_0$				
$5f_0$				

Table II. Square wave, duty factor 0.5. We ignore here the harmonics of even order, they should be very small (close to zero).

f	U_{RMS} [dBm]	U_{RMS}[mV]	U[mV]	A_k[mV]
f_0				
$3f_0$				
$5f_0$				
$7f_0$				
$9f_0$				

Table III. Triangle wave

f	U_{RMS} [dBm]	U_{RMS}[mV]	U[mV]	A_k[mV]
f_0				
$2f_0$				
$3f_0$				
$4f_0$				
$5f_0$				

Useful:

$$20\lg\frac{U_{RMS}[\mathrm{V}]}{U_{ref}} = U_{RMS}[\mathrm{dBm}]$$

The reference level is $U_{ref} = 223.6\mathrm{mV}$.

$$U_{RMS}[\mathrm{V}] = \frac{U[V]}{\sqrt{2}}$$

8.4 Make a graphical representation of the amplitude spectra: measured and computed. Compare the corresponding values.

8.5 Compute the distortion of the sine wave considering the RMS values from Table I and equation (16).

9. Presentation of Matlab environment

MATLAB (MATrix LABoratory) is an interactive program, developed by Math Works Inc.; its purpose is specially digital signal processing of data represented as vectors or matrix.
MATLAB integrates digital computation with the ability of vizualizing the results; as well as programming in a flexible environment.
MATLAB is an interactive system whose basic data element is an array that does not require dimensioning. It allows us to solve many technical computing problems, especially those with matrix and vector formulations, in a fraction of the time it would take to write a program in a scalar noninteractive language.
MATLAB has evolved over a period of years with input from many users. In university environments, it is the standard instructional tool for introductory and advanced courses in mathematics, engineering, and science. In industry, MATLAB is the tool of choice for high-productivity research, development, and analysis.

Working with MATLAB

After starting Matlab, a command window appears, that displays the prompter „ >> ” and waits for the user to enter a command. Execution of a command is usually followed by creating a variable in the workspace or by displaying a message and/or sketching a graph. Take for example the command:

>>*v*=0:10

This will create the variable *v* and will display its elements on the screen.

Besides working in the command window, MATLAB offers the possibility of working with M-files that contain code in the MATLAB language (with extension “.m”). There are two kinds of M-files:

* Scripts, which do not accept input arguments or return output arguments. They operate on data in the workspace.
* Functions, which can accept input arguments and return output arguments. Internal variables are local to the function.

Scripts

When invoking a script, MATLAB simply executes the commands found in the file. Scripts can operate on existing data in the workspace, or they can create new data on which to operate. Although scripts do not return output arguments, any variables that they create remain in the workspace, to be used in subsequent computations. In addition, scripts can produce graphical output using functions like plot.

Scripts do not allow their integration in other programs, realized on the principle of modularization. Script files are used to solve problems that require a long sequence of commands in a successive order that would normally be difficult to execute for iterative programming, such as in the command line.

Functions

The first line of a function M-file starts with the keyword *"function"*. It gives the function name and order of arguments. It differs from a script in the sense that it can accept input arguments, and can return output arguments. Variables defined inside the function are local to it; only the output variables remain.

Functions are used to extend MATLAB, to create new MATLAB commands. The general form of the first line in a function can be written as:

function [param_output] = name_funcion (param_input)

where:

function – keyword that declares the M-files as function (mandatory);

name_function – name of function, also the name of the M-file, without the m extension. It cannot be the same with a preexisting M-file name.

param_output – output parameters separated by comma and delimited by square brackets. If the functions has no output parameters, the square brackets and commas are removed.

param_input – input parameters separated by comma and delimited by square brackets. If the functions has no input parameters, the square brackets and commas are removed.

Signals

Unit impulse

Unit impulse is defined as follows:

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

Considering time shifting, we have:

$$\delta[n - n_0] = \begin{cases} 1, & n = n_0 \\ 0, & n \neq n_0 \end{cases}$$

In MATLAB we cannot define infinite length sequences; we have to define the range for the time n.

Define and plot in MATLAB the sequences:

1. $x_1[n] = \delta[n]$
2. $x_2[n] = \delta[n-1]$
3. $x_3[n] = \delta[n+1]$ for $-10 \le n \le 10$.

All vectors have 21 de elements.

x1.

```
% Generate unit impulse
clf; % erase old figure
n = -10:10; % generates a vector from  -10 to 10 (time)
d = [zeros(1,10) 1 zeros(1,10)]; % generates the unit impulse
stem(n,d); % graphical representation in discrete-time
xlabel('n');ylabel('Amplitude');
title('Unit Impulse');
axis([-10 10 0 1.2]);
```

x2.

```
n=-10:10;
x2=zeros(size(n));
x2(12)=1;
stem(n,x2),grid,title('x2[n]'),xlabel('n')
```

The moment of time *n* =1 corresponds to the 12-th element of the vector.

x3.

```
n=-10:10;
x3=zeros(size(n));
x3(10)=1;
stem(n,x3),grid,title('x3[n]'),xlabel('n')
```

The moment of time *n* =-1 corresponds to the 10-th element of the vector.

Exercises

Define and plot in MATLAB the sequences:

1. $x_1[n] = 0.7\delta[n-5]$ for $1 \le n \le 20$
2. $x_1[n] = 0.6\delta[n]$ for $-15 \le n \le 15$

Unit step

The unit step is defined as follows:

$$\sigma[n] = \begin{cases} 1, & n \ge 0 \\ 0, & n < 0 \end{cases}$$

Considering time-shifting, we have:

$$\sigma[n-n_0] = \begin{cases} 1, & n \ge n_0 \\ 0, & n < n_0 \end{cases}$$

In MATLAB we cannot define infinite length sequences; we have to define the range for the time n.

Define and plot in MATLAB the sequences:

1. $x_1[n] = \sigma[n]$ for $-10 \le n \le 10$.

2. $x_2[n] = \sigma[n-2]$ for $-5 \le n \le 10$.
3. $x_3[n] = \sigma[n+2]$ for $-5 \le n \le 10$.

x1.

```
% Generate unit step
n = -10:10; % generates a vector from  -10 to 10 (time)
u = [zeros(1,10) ones(1,11)]; % generates the unit step
stem(n,u); % graphical representation in discrete-time
xlabel('n');ylabel('Amplitude');
title('Unit step');
axis([-10 10 0 1.2]);
```

x2.

```
n=-5:10;
x2=[zeros(1,7),ones(1,9)];
stem(n,x2),grid,title('x_2[n]'),xlabel('n')
```

The moment of time $n = 2$ corresponds to the 18-th element of the vector.

x3.

```
n=-5:10;
x3=[zeros(1,3),ones(1,13)];
stem(n,x3),grid,title('x_3[n]'),xlabel('n')
```

The moment of time $n = -2$ corresponds to the 4-th element of the vector.

Exercises

Define and plot in MATLAB the sequences:

1. $x_1[n] = 0.7\sigma[n]$ for $-10 \le n \le 20$
2. $x_2[n] = \sigma[n-7]$ for $0 \le n \le 30$
3. $x_3[n] = 1.8\sigma[n+3]$ for $-15 \le n \le 15$