Transformarea *z*

http://shannon.etc.upt.ro/teaching/ps/Cap10_TransfZ.pdf

Transformarea z generalizeaza transformarea Fourier in timp discret pentru intreg planul complex la fel ca si transformarea Laplace care generalizează transformarea Fourier pentru intreg planul complex.
Pentru variabila complexa se foloseste notatia:

$$z = x + j \cdot y = r \cdot e^{j\Omega}; \qquad x,\ y \in \mathbb{R}$$

$$|z| = r = \sqrt{x^2 + y^2}$$

$$\Omega = \arg\{z\}$$

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intrare $\quad x[n] = z_0^n = r_0^n e^{j\Omega_0 n} \longrightarrow$ Sistem $h[n]$ $\longrightarrow$ iesire $\quad y[n] = h[n] * z_0^n = z_0^n \cdot H(z_0)$

Raspunsul unui sistem discret liniar si invariant in timp la exponentiala complexa discreta de modul neunitar

$$z = x + jy = re^{j\Omega}\ ;\ x,y \in R;\ |z| = r = \sqrt{x^2 + y^2} \ \text{ si } \Omega = arg\{z\}$$

$$z_0^n = r_0^n \cdot e^{j\Omega_0 n}\ ;\ y[n] = h[n] * z_0^n = \sum_{k=-\infty}^{\infty} h[k] \cdot z_0^{n-k} = z_0^n \cdot \sum_{k=-\infty}^{\infty} h[k] \cdot z_0^{-k}$$

$$H(z) = \sum_{k=-\infty}^{\infty} h[k] \cdot z^{-k}\ ;\ y[n] = z_0^n \cdot H(z_0)$$

$$x[n] = \sum_k c_k z_k^n \ \Rightarrow\ y[n] = \sum_k c_k H(z_k) z_k^n.$$

Pentru orice sistem discret LIT

-functie proprie z_0^n

-valoare proprie $H(z_0)$

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Transformarea *z* bilaterala

Transformarea *z* bilaterala a semnalului *x*[*n*] discret aperiodic:

$$\mathcal{Z}\{x[n]\}(z) = X(z) = \sum_{n=-\infty}^{\infty} x[n] \cdot z^{-n} \;; z = r \cdot e^{j\Omega} \;, r \geq 0.$$

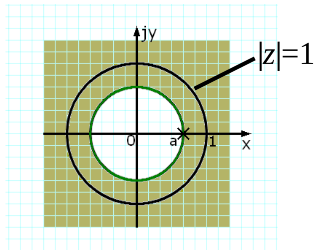
Legatura dintre transformata Z si transformata Fourier in timp discret:

$$\mathcal{Z}\{x[n]\}(z) = \sum_{n=-\infty}^{\infty} \left(x[n] \cdot r^{-n}\right) \cdot e^{-j\Omega n} = \mathcal{F}\{r^{-n} \cdot x[n]\}(\Omega) \;; r \geq 0.$$

$$|z| = r = 1,\; \mathcal{Z}\{x[n]\}\left(e^{j\Omega}\right) = \mathcal{F}\{x[n]\}(\Omega).$$

$$|X(z)| \leq \sum_{n=-\infty}^{\infty} \left|r^{-n} x[n]\right| \cdot \left|e^{-j\Omega n}\right| = \sum_{n=-\infty}^{\infty} \left|r^{-n} x[n]\right| < \infty$$

$$r^{-n} x[n] \in l^1.$$



Transformata z evaluata pe cercul unitar, de raza |*z*|=1, este transformata Fourier in timp discret

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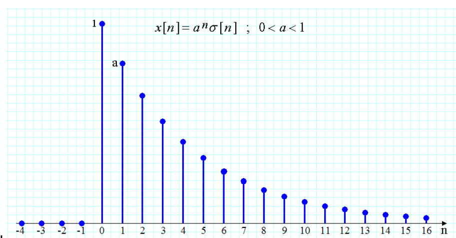
Exemple

$$1.\; x[n] = a^n \sigma[n],\; |a| < 1.$$

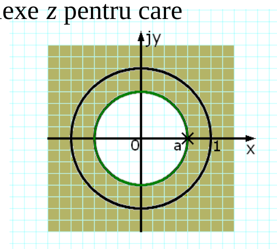


$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} \left(az^{-1}\right)^n$$

$$\text{Daca } \left|az^{-1}\right| < 1 \Rightarrow X(z) = \frac{1}{1 - az^{-1}},\; |z| > |a|.$$

$$X(\Omega) = \frac{1}{1 - ae^{-j\Omega}}.$$

Regiunea de convergenta (ROC) = valorile variabilei complexe *z* pentru care *X*(*z*) exista



X(z)-functie rationala, **zerourile** = radacini ale numaratorului; **polii** = radacini ale numitorului.

Constelatia de poli si zerouri (PZC).

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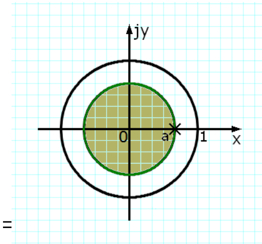


2. $x[n] = -a^n \sigma[-n-1]$

$$X(z) = -\sum_{n=-\infty}^{-1} a^n z^{-n} = -\sum_{n=-\infty}^{-1} \left(az^{-1}\right)^n = -\sum_{n=1}^{\infty} \left(\frac{1}{az^{-1}}\right)^n =$$

$$= -\frac{1}{az^{-1}} \sum_{n=0}^{\infty} \left(\frac{1}{a \cdot z^{-1}}\right)^n$$

Daca $\frac{1}{|a||z^{-1}|} < 1$ atunci $X(z) = -\frac{1}{az^{-1}\left(1 - \frac{1}{az^{-1}}\right)} =$

$$= \frac{1}{1 - az^{-1}}; \ |z| < |a|.$$

Proprietatile domeniului de convergenta al transformarii *z* bilaterale

1. Domeniul de convergenta al transformatei z bilaterale nu poate contine nici un pol al acesteia.

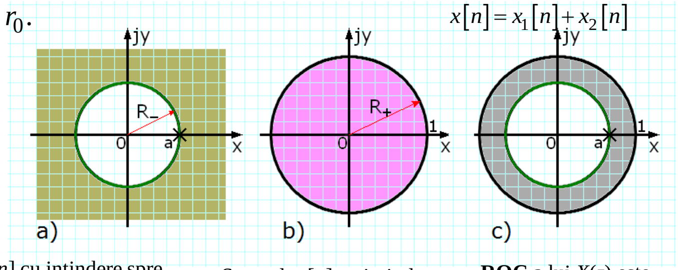
2. Daca $x[n]$ este un semnal cu suportul de durata finita, $X(z)$, transformata sa bilaterala exista in tot planul, cu exceptia evenual a punctelor $z = 0$ si/sau $z = \infty$.

3. Daca $x[n]$ este un semnal cu intindere infinita spre dreapta si daca cercul $|z| = r_0$ este din domeniul de convergenta al transformatei z bilaterale, atunci toate punctele din plan $|z| > r_0$, cu exceptia eventual a punctului de la infinit, sunt din domeniul de convergenta.

4. Daca $x[n]$ este un semnal cu intindere infinita spre stanga si daca cercul $|z| = r_0$, este inclus in domeniul de convergenta al transformatei z bilaterale atunci toate punctele din plan $|z| \le r_0$, cu exceptia eventual a punctului din origine sunt din domeniul de convergenta.

5. Daca $x[n]$ este un semnal cu intindere infinita atat la stanga cat si la dreapta si daca cercul $|z| = r_0$ este inclus in domeniul de convergenta al transformatei z bilaterale, atunci acesta este o coroana circulara ce include cercul $|z| = r_0$.

$x[n] = x_1[n] + x_2[n]$



Semnal $x_1[n]$ cu intindere spre dreapta (cauzal); **ROC** in afara cercului de raza R^-

Semnal $x_2[n]$ cu intindere spre stanga (anti-cauzal); **ROC** disc cu raza R^+

ROC a lui $X(z)$ este intersectia celor doua ROC (cerc)

Exemplu

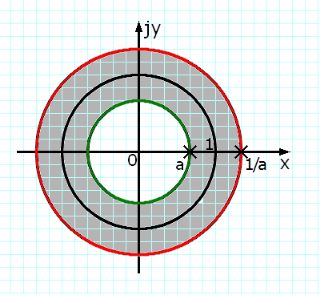
3. $x[n] = a^{|n|}$, $0 < a < 1$, $x_1[n] = a^n \sigma[n]$,
$x_2[n] = a^{-n}\sigma[-n-1]$, $x[n] = x_1[n] + x_2[n]$.

$$a^n\sigma[n] \to \frac{1}{1-az^{-1}};\ |z| > a.$$

$$a^{-n}\sigma[-n-1] = \left(\frac{1}{a}\right)^n \sigma[-n-1] \to$$

$$\to \frac{-1}{1-\frac{1}{a}z^{-1}},\ |z| < \frac{1}{a}.$$

$$a^{|n|} \to \frac{a^2-1}{a}\frac{z}{(z-a)\left(z-\frac{1}{a}\right)};\ a < |z| < \frac{1}{a}.$$



Trasformarea z inversa

Fie Γ un contur inchis inclus in domeniul de convergenta al transformatei z bilaterale

$\sum_{k=-\infty}^{\infty} x[k]\cdot z^{-k}$ se inmulteste cu z^{n-1} si se integreaza pe cercul Γ:

$\int_\Gamma z^{n-1}dz = \int_\Gamma \sum_{k=-\infty}^{\infty} x[k]\, z^{n-k-1}dz = \sum_{k=-\infty}^{\infty} x[k]\int_\Gamma \frac{dz}{z^{k-n+1}}$. Pentru orice functie olomorfa in DC, $f(z)$ si orice curba inchisa Γ din DC avem: $\int_\Gamma \frac{f(z)}{(z-a)^{n+1}}dz = \frac{2\pi j}{n!}\frac{d^n}{dz^n} f(z)$,

a un punct interior curbei. Functia $f(z)=1$ este olomorfa in intreg planul complex, deci si in DC. Punctul $a=0,$ este interior cercului Γ considerat.

$$\int_\Gamma \frac{f(z)}{(z-a)^{n+1}}dz = \begin{cases} \frac{2\pi j}{0!}\cdot 1, & n=k \\ 0, & n\neq k \end{cases} \Rightarrow x[n] = \frac{1}{2\pi j}\int_\Gamma X(z)z^{n-1}dz \;;\; \Gamma\in \mathrm{DC}$$

$$Z^{-1}\{X(z)\}[n] = \frac{1}{2\pi j}\int_\Gamma X(z)z^{n-1}dz = x[n] \;\;;\;\; x[n]\overset{z}{\leftrightarrow} X(z).$$

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Definirea transformatei z bilaterale prin constelatia de poli si zerouri

Daca $X(z)$ este o fractie rationala:

$$X(z) = k\frac{\prod_{k=1}^{M}(z-z_{0k})}{\prod_{k=1}^{N}(z-z_{pk})} \;;\; z\in DC.$$

Este suficienta cunoasterea polilor z_{pk} si a zerourilor z_{0k} pentru a cunoaste $X(z)$ cu exceptia constantei k.

Daca se cunoaste in plus si valoarea $X(z_0)$ intr-un anumit punct z_0 din DC, se poate determina si constanta k.

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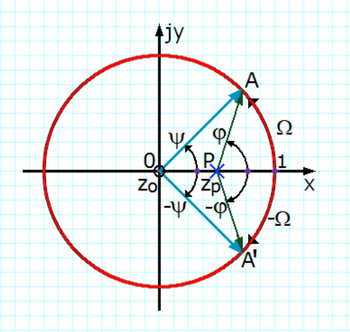
Exemplu



Fie constelatia de poli si zerouri din figura cu $z_0 = 0$ si $z_p = 0,5$.

Pentru $k = 1$: $X(z) = \dfrac{z}{z - 0,5} = \dfrac{1}{1 - 0,5 z^{-1}}$.

Daca semnalul $x[n]$ este cauzal atunci

DC : $|z| > |z_p| = 0,5$.

Deoarece cercul unitar este in interiorul DC semnalul $x[n]$ are si transformata Fourier in timp discret:

$$X(\Omega) = \frac{e^{j\Omega}}{e^{j\Omega} - 0,5} = \frac{1}{1 - 0,5 e^{-j\Omega}}$$

Fie punctul A de pe cercul unitar, plasat la unghiul Ω (lungimea arcului de cerc in radiani).

$$|X(\Omega)| = \frac{\left|\overline{OA}\right|}{\left|\overline{AP}\right|} \text{ ; } \Phi(\Omega) = \arg X(\Omega) = \psi - \varphi.$$

In cazul general, notand cu z_{pk} si cu z_{0k} polii si zerourile si cu ψ_k si φ_k unghiurile pe care le formeaza vectorii Az_{pk} si Az_{0k} cu axa orizontala, se poate scrie:

$$|X(z)| = |k| \frac{\prod_{k=1}^{M} \left|\overline{Az_{0k}}\right|}{\prod_{k=1}^{N} \left|\overline{Az_{pk}}\right|} \text{ ; } \Phi(\Omega) = \arg k + \sum_{k=1}^{M} \psi_k - \sum_{k=1}^{N} \varphi_k .$$

Frecventa Ω reprezinta lungimea arcului de cerc masurat in radiani, pe cercul unitar, de la intersectia acestuia cu semiaxa $x > 0$ si pana in punctul A, sensul fiind cel trigonometric.

Transformata *z* unilaterala

$$Z_u\{x[n]\}(z) = X_u(z) = \sum_{n=0}^{\infty} x[n]z^{-n}.$$

Transformata z unilaterala poate fi privita ca si transformata z bilaterala a unui **semnal cauzal**.

Domeniul de convergenta al transformatei unilaterale este fie tot planul, fie exteriorul unui disc cu centrul in origine.

Transformata z unilaterala este indicata pentru studiul sistemelor cauzale caracterizate de ecuatii cu diferente finite liniare si cu coeficienti constanti care au **conditii initiale nenule.**

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Proprietatile transformarii *z*

Notatii:

$$x[n] \overset{Z}{\leftrightarrow} X(z)\ ,\ z \in DC_x\ ;\ x[n] \overset{Z_u}{\leftrightarrow} X_u(z)$$

$$y[n] \overset{Z}{\leftrightarrow} Y(z)\ ,\ z \in DC_y\ ;\ y[n] \overset{Z_u}{\leftrightarrow} Y_u(z).$$

Tema de curs: Demonstratiile

1. Liniaritate

$$ax[n] + by[n] \leftrightarrow aX(z) + bY(z)\ ;\ z \in DC_x \cap DC_y\ ,\ \text{cel putin}$$

$$ax[n] + by[n] \leftrightarrow aX_u(z) + bY_u(z)$$

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2. Translatia in timp

$x[n-n_0] \leftrightarrow z^{-n_0} X(z)$, $z \in DC.$

Demonstratie

$$Z\{x[n-n_0]\} = \sum_{n=-\infty}^{\infty} x[n-n_0] z^{-n} \overset{n-n_0=m}{=} \sum_{m=-\infty}^{\infty} z^{-(m+n_0)} = z^{-n_0} \sum_{m=-\infty}^{\infty} x[m] z^{-m}$$

Daca $n_0 > 0,\ z = 0$ se elimina din DC iar daca $n_0 < 0$, $z = \infty$ se elimina din DC.

In cazul transformarii unilaterale, pentru $n_0 > 0$:

$$Z_u\{x[n-n_0]\} = \sum_{n=0}^{\infty} x[n-n_0] z^{-n} \overset{n-n_0=m}{=} \sum_{m=-n_0}^{\infty} x[m] z^{-m} z^{-n_0}$$

$$= z^{-n_0} \left(\sum_{m=0}^{\infty} x[m] z^{-m} + \sum_{m=-n_0}^{-1} x[m] z^{-m} \right),$$

$$x[n-n_0] \leftrightarrow z^{-n_0} \left(X_u(z) - \sum_{n=-n_0}^{-1} x[n] z^{-n} \right)$$

3. Modularea in timp

$e^{j\Omega_0 n} x[n] \leftrightarrow X\left(e^{-j\Omega_0} z\right)$, $z \in DC.$

Demonstratie

$$Z\left\{e^{j\Omega_0 n} x[n]\right\} = \sum_{n=-\infty}^{\infty} e^{j\Omega_0 n} x[n] z^{-n} = \sum_{n=-\infty}^{\infty} x[n] \left(e^{-j\Omega_0} z\right)^{-n},$$

$$e^{j\Omega_0 n} x[n] \leftrightarrow X_u\left(e^{-j\Omega_0} z\right).$$

Mai general:

$$z_0^n x[n] \leftrightarrow X\left(\frac{z}{z_0}\right) \qquad \frac{z}{z_0} \in \mathrm{DC}; \quad z_0^n x[n] \leftrightarrow X_u\left(\frac{z}{z_0}\right).$$

4. Reflectarea semnalului

$x[-n] \leftrightarrow X\left(z^{-1}\right)$, $\frac{1}{z} \in DC.$

Demonstratie

$$Z\{x[-n]\} = \sum_{n=-\infty}^{\infty} x[-n] z^{-n} \overset{m=-n}{=} \sum_{m=-\infty}^{\infty} x[m] \left(\frac{1}{z}\right)^{-m} = X\left(\frac{1}{z}\right).$$

5. Diferentierea in domeniul n

$$x[n]-x[n-1]\leftrightarrow\left(1-z^{-1}\right)X(z)\ ;\ z\in DC\ ;$$

$$x[n]-x[n-1]\leftrightarrow\left(1-z^{-1}\right)X_u(z)-x[-1].$$

6. Insumarea in domeniul n

$$\sum_{k=-\infty}^{n} x[k]\leftrightarrow\frac{X(z)}{1-z^{-1}}=\frac{z}{z-1}X(z)\ ;\ z\in \mathrm{DC}\cap\mathrm{DC}^*;\ \mathrm{DC}^*=\left\{z\,\middle|\,|z|>1\right\}.$$

$$\sum_{k=-\infty}^{n} x[k]\leftrightarrow\frac{X_u(z)+\sum\limits_{k=-\infty}^{-1}x[k]}{1-z^{-1}}\ ;\ |z|>1.$$

Demonstratie

Fie $y[n]=\sum\limits_{k=-\infty}^{m}x[k]\ \Rightarrow\ x[n]=y[n]-y[n-1]\leftrightarrow X(z)=\left(1-z^{-1}\right)Y(z);$

$$Y(z)=\frac{1}{1-z^{-1}}X(z).$$

Pentru transformata unilaterala:

$$x[n]=y[n]-y[n-1]\leftrightarrow X_u(z)=\left(1-z^{-1}\right)Y(z)-y[-1],\ y[-1]=\sum_{k=-\infty}^{-1}x[k].$$

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7. Diferentierea in domeniul z

$$nx[n]\leftrightarrow -z\frac{dX(z)}{dz}\ ;\ z\in\mathrm{DC}.$$

$$nx[n]\leftrightarrow -z\frac{dX_u(z)}{dz}.$$

8. Conjugarea complexa in domeniul timp

$$x^*[n]\leftrightarrow X^*\left(z^*\right)\ ,\ z\in\mathrm{DC}.$$

$$x^*[n]\leftrightarrow X_u^*\left(z^*\right).$$

Demonstratie

$$Z\left\{x^*[n]\right\}=\sum_{n=-\infty}^{\infty}x^*[n]z^{-n}=\left(\sum_{n=-\infty}^{\infty}x[n]\left(z^*\right)^{-n}\right)^*=X^*\left(z^*\right);$$

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9. Teorema convolutiei in domeniul timp

$x[n], y[n] \in C.$

$x[n] * y[n] \leftrightarrow X(z)Y(z) \quad z \in \mathrm{DC}_x \cap \mathrm{DC}_y$ cel putin.

$x[n] * y[n] \leftrightarrow X_u(z)Y_u(z).$

Demonstratie

$$Z\{x[n] * y[n]\} = \sum_{n=-\infty}^{\infty} (x[n] * y[n]) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x[k] y[n-k] z^{-k} z^{-(n-k)}$$

$$\overset{n-k=m}{=} \sum_{m=-\infty}^{\infty} y[m] z^{-m} \sum_{k=-\infty}^{\infty} x[k] z^{-k} = Y(z)X(z).$$

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10. Teorema produsului semnalelor in domeniul timp

$$x[n]y[n] \leftrightarrow \frac{1}{2\pi j} \int_\Gamma X(u) Y\left(\frac{z}{u}\right) \frac{du}{u}, \ \Gamma \in \mathrm{DC}.$$

$$\mathrm{DC}_x = \left\{ z \middle| R_x^- < |z| < R_x^+ \right\}; \ \mathrm{DC}_y = \left\{ z \middle| R_y^- < |z| < R_y^+ \right\}$$

$$\Rightarrow R_x^- < |u| < R_x^+ \text{ si } R_y^- < \frac{|z|}{|u|} < R_y^+.$$

Prin inmultire $\Rightarrow$

$$R_x^- \cdot R_y^- < |z| < R_x^+ \cdot R_y^+ \ \Leftrightarrow \mathrm{DC} = \left\{ z \middle| R_x^- \cdot R_y^- < |z| < R_x^+ \cdot R_y^+ \right\}.$$

$$x[n]y[n] \leftrightarrow \frac{1}{2\pi j} \int_\Gamma X_u(u) Y_u\left(\frac{z}{u}\right) \frac{du}{u}, \ \Gamma \in \mathrm{DC}.$$

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Demonstratie

$$Z\left\{x[n]y[n]\right\}=\sum_{n=-\infty}^{\infty}x[n]y[n]z^{-n}=\sum_{n=-\infty}^{\infty}\left(\frac{1}{2\pi j}\int_{\Gamma}X(u)u^{n-1}du\right)y[n]z^{-n}=$$

$$=\frac{1}{2\pi j}\left[\int_{\Gamma}X(u)\sum_{n=-\infty}^{\infty}y[n]\left(\frac{z}{u}\right)^{-n}\right]\frac{du}{u}=\frac{1}{2\pi j}\int_{\Gamma}X(u)Y\left(\frac{z}{u}\right)\frac{du}{u}.$$

Particularizari.

$$x[n]\in l^2.\ y[n]=x^*[n]\Rightarrow\left|x[n]\right|^2\leftrightarrow\frac{1}{2\pi j}\int_{\Gamma}X(u)\,X^*\left(\frac{z^*}{u^*}\right)\frac{du}{u}\Rightarrow$$

$$\Rightarrow\sum_{n=-\infty}^{\infty}\left|x[n]\right|^2z^{-n}=\frac{1}{2\pi j}\int_{\Gamma}X(u)\,X^*\left(\frac{z^*}{u^*}\right)\frac{du}{u}.$$

Daca $u=e^{j\Omega}\Rightarrow u^*=e^{-j\Omega}$ si $du=je^{j\Omega}d\Omega$ si

$$\sum_{n=-\infty}^{\infty}\left|x[n]\right|^2=\frac{1}{2\pi j}\int_{-\pi}^{\pi}X\left(e^{j\Omega}\right)X^*\left(e^{j\Omega}\right)j\frac{e^{j\Omega}d\Omega}{e^{j\Omega}}=$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|X\left(e^{j\Omega}\right)\right|^2d\Omega=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|X(\Omega)\right|^2d\Omega$$

11. Teorema valorii initiale aunui semnal discret cauzal

$$x[0]=\lim_{|z|\to\infty}X_u(z)=\lim_{|z|\to\infty}X(z).$$

Demonstratie

$$X(z)=X_u(z)=\sum_{n=0}^{\infty}x[n]z^{-n}=x[0]+\frac{1}{z}x[1]+\ldots$$

Trecand la limita se obtine relatia din enunt.

12. Teorema valorii finale a unui semnal discret cauzal

$$x[\infty]=\lim_{z\to1}(z-1)X_u(z)=\lim_{z\to1}(z-1)X(z).$$

Demonstratie

$$\sum_{n=0}^{\infty}\left(x[n+1]-x[n]\right)z^{-n}=\sum_{n=0}^{\infty}x[n+1]z^{-n}-X_u(z)\overset{n+1=m}{=}\sum_{m=1}^{\infty}x[m]z^{-(m-1)}-X_u(z)=$$

$$=z\left(\sum_{m=0}^{\infty}x[m]z^{-m}-x[0]\right)-X_u(z)=(z-1)X_u(z)-zx[0].$$

Pentru $z\to1$: $\lim_{z\to1}\sum_{n=0}^{\infty}\left(x[n+1]-x[n]\right)z^{-n}=\lim_{z\to1}(z-1)X_u(z)-x[0]$,

sau: $x[\infty]-x[0]=\lim_{z\to1}(z-1)X_u(z)-x[0]\ \Leftrightarrow x[\infty]=\lim_{z\to1}(z-1)X_u(z).$

Relatia dintre transformata *z* si transformata Laplace

$x_a(t) \leftrightarrow X_a(s)$ Prin esantionare se obtine

$$\hat{x}(t) = \sum_{n=-\infty}^{\infty} x_a(nT_s)\delta(t-nT_s) \Rightarrow$$

$$\Rightarrow \mathcal{L}\{\hat{x}(t)\} = \sum_{n=-\infty}^{\infty} x_a(nT_e)\mathcal{L}\{\delta(t-nT_s)\} = \sum_{n=-\infty}^{\infty} x_a(nT_e)e^{-snT_s}.$$

$$x_a(nT_s) = x_d[n] \Rightarrow Z\{x_d[n]\} = \sum_{n=-\infty}^{\infty} x_d[n]z^{-n} = \sum_{n=-\infty}^{\infty} x_a(nT_s)z^{-n}$$

$$\mathcal{L}\{x_a(t)\delta_{T_s}(t)\}\Big|_{e^{sT_s}=z} = Z\{x_d[n]\}; \quad \mathcal{L}_u\{x_a(t)\delta_{T_s}(t)\}\Big|_{e^{sT_s}=z} = Z_u\{x_d[n]\}.$$

Tr.Laplace a semnalului analogic esantionat Tr.Z a semnalului digital corespunzator

$$\mathcal{L}\{x_a(t)\delta_{T_s}(t)\}\Big|_{e^{sT_s}=z} = Z\{x_d[n]\}; \quad x_d[n] = x_a(nT_s)$$

$$z = e^{sT_s}$$

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Studiul sistemelor discrete liniare si invariante in timp prin intermediul transformarii *z*

Forma simpla a teoremei convolutiei semnalelor discrete face din transformata *Z* un instrument util pentru studiul sistemelor discrete LIT.

Daca sistemul studiat este cauzal si are conditii initiale **nenule**, este utila transformarea *Z* unilaterala.

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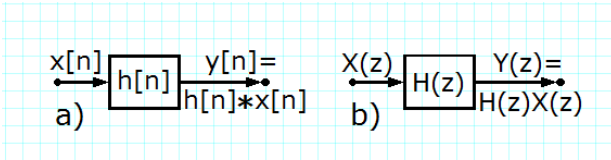
Functia de sistem a unui sistem discret, liniar si invariant in timp



Functia $H(z)$, numita "functia (de) sistem" sau "functia de transfer a sistemului" caracterizeaza complet comportarea sa în planul complex z, dupa cum raspunsul la impuls $h[n]$ caracterizeaza complet comportarea sistemelor in timp.

Daca sistemul discret este **stabil**, exista si transformata Fourier in timp discret a lui $h[n]$ si prin urmare **cercul unitar este in domeniul de convergenta** al functiei de sistem ce caracterizeaza un sistem stabil.

Daca sistemul este **cauzal:** $\boldsymbol{H_u(z) = H(z)}$**,** atunci domeniul de convergenta al functiei este exteriorul unui disc.

Daca sistemul este **stabil si cauzal, cercul unitar este in DC.** Toti **polii** functiei sistem a unui sistem stabil si cauzal au **modulul subunitar** ceea ce este ehivalent cu faptul ca sunt cuprinsi în interiorul discului unitar:

$$|z_p| < 1$$

Determinarea raspunsului unui sistem discret liniar si invariant în timp, utilizand transformarea Z

Daca $x[n]$ este dat si se specifica $h[n] \leftrightarrow H(z)$ atunci se aplica semnalului de intrare transformarea directa rezultand $X(z)$. Raspunsul in complex este $Y(z)=H(z)X(z)$.

Aplicand transformarea Z inversa, rezulta semnalul raspuns, $y[n]$. Daca se lucreaza cu sisteme si semnale cauzale în conditii initiale nenule, se va utiliza numai transformarea Z unilaterala.

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Calculul transformarii *z* inverse

Sunt aplicabile trei metode:

1. **Calculul direct al integralei,**
2. **Transformarea functiei $Y(z)$ intr-o suma de fractii simple,**
3. **Metoda dezvoltarii functiei $Y(z)$ in serie de puteri.**

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2. Transformarea functiei *Y*(*z*) intr-o suma de fractii simple

Metoda se aplica in cazurile in care $Y(z)$ este o functie rationala. Functia $Y(z)$ poate fi un raport de polinoame în z^{-1} sau z. Recomandam sa se lucreze în z^{-1}, notand

pentru comoditate $z^{-1} = x$, deoarece in majoritatea cazurilor tabelele sunt date în functie de puterile lui z^{-1}.

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$$Y(x) = Y\left(z^{-1}\right) = \frac{N(x)}{D(x)} = I(x) + \frac{R(x)}{D(x)}.$$

$$I(x) = \sum_k c_k x^k = \sum_k c_k z^{-k} \leftrightarrow \sum_k c_k \delta[n-k]$$

$$\frac{R(x)}{D(x)} = \sum_m \frac{a_m}{x - x_m} + \sum_k \sum_{i=1}^{s_k} \frac{b_{ki}}{(x - x_k)^i}.$$

$$a_m = \left[(x - x_m) \frac{R(x)}{D(x)} \right]\Bigg|_{x = x_m};$$

$$b_{ki} = \frac{1}{(s_k - i)!} \left\{ \frac{d^{s_k - i}}{dx^{s_k - i}} \left[(x - x_k)^{s_k} \frac{R(x)}{D(x)} \right] \right\}\Bigg|_{x = x_k}.$$

Se aplica transformarea inversa fiecarui termen din suma, dupa ce se revine mai intai la $z^{-1} = x$, utilizand tabelele de transformate si tabelele de proprietati ale transformarii.

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$$\frac{R(x)}{D(x)} = \sum_m \frac{a_m}{x - x_m} + \sum_k \sum_{i=1}^{s_k} \frac{b_{ki}}{(x - x_k)^i}$$

$$a_m = \left[(x - x_m) \frac{R(x)}{D(x)} \right] \Big|_{x = x_m} \qquad b_{ki} = \frac{1}{(s_k - i)!} \left\{ \frac{d^{s_k - i}}{dx^{s_k - i}} \left[(x - x_k)^{s_k} \frac{R(x)}{D(x)} \right] \right\} \Big|_{x = x_k}$$

Exemple

1. $Y(z) = \dfrac{z}{(z-0.5)(z-0.25)} = Y_u(z) \Rightarrow \mathrm{DC} = \{z \,|\, |z| > 0.5\}$

$$Y(z) = \frac{8z^{-1}}{(2-z^{-1})(4-z^{-1})} = \frac{8x}{(2-x)(4-x)} = \frac{8}{2-x} - \frac{16}{4-x};$$

$$Y(z) = \frac{8}{2-z^{-1}} - \frac{16}{4-z^{-1}} = \frac{4}{1-0,5z^{-1}} - \frac{4}{1-0,25z^{-1}}.$$

$$y[n] = 4\left[(0,5)^n - (0,25)^n\right]\sigma[n].$$

2. Aceeasi transformata $Y(z)$ dar $\mathrm{DC} = \{z \,|\, |z| < 0,25\}$

$$y[n] = -4\left[(0,5)^n - (0,25)^n\right]\sigma[-n-1].$$

3. Aceeasi transformata $Y(z)$ dar $\mathrm{DC} = \{z \,|\, 0,25 < |z| < 0,5\}$

$$y[n] = -4(0,5)^n \sigma[-n-1] - 4(0,25)^n \sigma[n].$$

3. Metoda dezvoltarii functiei *Y* (*z*) in serie de puteri

Dezvoltarea functiei $Y(z)$ in serie de puteri in jurul originii

Exemplul #a

$$Y(z) = e^{z^{-1}}, \quad ROC: |z| > 0$$

$$e^{z^{-1}} = 1 + \frac{1}{1!}z^{-1} + \frac{1}{2!}z^{-2} + \ldots + \frac{1}{n!}z^{-n} + \ldots = \sum_{n=0}^{\infty} \frac{1}{n!}z^{-n}$$

$$\Rightarrow y[n] = \frac{1}{n!}$$

Exemplul #b

$$Y(z) = e^{z}, \quad ROC: |z| \geq 0, |z| \neq \infty$$

$$e^{z} = 1 + \frac{1}{1!}z + \frac{1}{2!}z^{2} + \ldots + \frac{1}{m!}z^{m} + \ldots = \sum_{m=0}^{\infty} \frac{1}{m!}z^{m}$$

$$= \sum_{n=-\infty}^{0} \frac{1}{(-n)!}z^{-n} = 1 + \sum_{n=-\infty}^{-1} \frac{1}{(-n)!}z^{-n}$$

$$\Rightarrow y[n] = \delta[n] - \frac{1}{(-n)!}\sigma[-n-1] = \frac{1}{(-1)^{n} n!}\sigma[-n]$$

Exemplul #c

$$Y(z) = \ln\left(1 + az^{-1}\right), \quad ROC: |z| > |a| \Rightarrow \textbf{semnal cauzal}$$

$$Y(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} a^n}{n} z^{-n} \longrightarrow y[n] = \frac{(-1)^{n+1} a^n}{n}; \quad n \geq 1$$

Teorema valorii initiale:

$$y[0] = \lim_{z \to \infty} \ln\left(1 + az^{-1}\right) = 0$$

$$y[n] = \frac{(-1)^n}{n} a^n \sigma[n-1]$$

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Exemplul #d

$$Y(z) = \frac{1}{1 - az^{-1}}, \quad |z| > a \quad \Rightarrow \quad y[n] = a^n \sigma[n]$$

$$\begin{array}{l|l}
1 & 1 - az^{-1} \\
\hline
\underline{-1 + az^{-1}} & 1 + az^{-1} + a^2z^{-2} + a^3z^{-3} + \ldots \\
/ \quad az^{-1} & \\
\quad \underline{-az^{-1} + a^2z^{-2}} & \\
\quad / \quad a^2z^{-2} & \\
\qquad \underline{-a^2z^{-2} + a^3z^{-3}} & \\
\qquad / \quad a^3z^{-3} & \\
\qquad \qquad \ddots &
\end{array}$$

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Exemplul #e, aceeasi transformata ca si la #d, ROC diferita

$$Y(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}, \quad |z| < |a|$$

$\Rightarrow$ semnal anticauzal, transformata z contine doar puteri ale lui z

$$\Rightarrow Y(z) = -\sum_{m=1}^{\infty} \left(a^{-1}z\right)^m = -\sum_{n=-\infty}^{-1} a^n z^{-n} \quad \Rightarrow \quad y[n] = -a^n \sigma[-n-1]$$

$$\begin{array}{l|l}
z & -a + z \\
\hline
-z + a^{-1}z^2 & -a^{-1}z - a^{-2}z^2 - a^{-3}z^3 - a^{-4}z^4 \ldots \\
\hline
/ \quad a^{-1}z^2 & \\
\quad -a^{-1}z^2 + a^{-2}z^3 & \\
\hline
\quad / \quad a^{-2}z^3 & \\
\qquad -a^{-2}z^3 + a^{-3}z^4 & \\
\hline
\qquad / \quad a^{-3}z^4 & \\
\qquad\qquad \ddots &
\end{array}$$

Sisteme discrete liniare si invariante in timp, caracterizate prin ecuatii cu diferente finite liniare si cu coeficienti constanti

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]; \quad a_0 \neq 0$$

$$\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z) \quad \Rightarrow \quad H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} = \frac{N(z)}{D(z)} \quad \text{functia de transfer}$$

Functia de transfer a unui sistem discret LIT: functie rationala in z or z^{-1}.

Zerouri : radacinile numaratorului $N(z)$;

Poli : radacinile numitorului $D(z)$.

Pentru un sistem c**auzal si stabil: polii sunt in interiorul cercului unitar** |z|=1 $\left|z_{pk}\right|<1$

Teorema valorii initiale:

$$h[0]=\lim_{z\to\infty} H_u(z)=\lim_{z\to\infty} H(z)=\lim_{z\to\infty} \frac{N(z)}{D(z)} \text{ - finita}$$

Gr. M

Gr. N

Gradul numaratorului functiei de transfer a unui sistem cauzal si stabil este mai mic sau egal cu gradul numitorului, M≤N

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Contributia polilor unui sistem discret cauzal in raspunsul la impuls al acestuia

Vom considera numai cazul polilor simpli si dubli, celelalte cazuri tratandu-se in mod identic.

1. Perechi de poli simpli complex conjugati; $z_p = r_p e^{j\Omega_p}$ si $z_p^* = r_p e^{-j\Omega_p}$

Termenii corespunzatori in descompunerea lui $H(z)$ in fractii simple sunt:

$\ldots+\frac{a}{1-r_p e^{j\Omega_p} z^{-1}}+\frac{a^*}{1-r_p e^{-j\Omega_p} z^{-1}}+\ldots$ iar contributia lor in $h[n]$ este de forma

$A\, r_p^n \sin\left(\Omega_p n+\Phi_n\right)$.

Daca $r_p<1$, raspunsul se amortizeaza, perechea de poli nu confera instabilitate sistemului.

Daca $r_p>1$, raspunsul creste exponential, sistem-instabil.

Caz aparte: poli simpli situati pe cercul unitar. Contributia lor la $h[n]$ este de forma:

$A\sin\left(\Omega_p n+\Phi_p\right)$, oscilatie de amplitudine fixa (chiar si dupa disparitia excitatiei).

Stabilitate la limita-oscilator.

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2. Pereche de poli dubli complex conjugati,

$z_p = r_p e^{j\Omega_p}$ si $z_p^* = r_p e^{-j\Omega_p}$

Contributia lor la functia de transfer este de forma:

$$\ldots + \frac{a_1}{1 - r_p e^{j\Omega_p} z^{-1}} + \frac{a_1^*}{1 - r_p e^{-j\Omega_p} z^{-1}} + \frac{a_2}{\left(1 - r_p e^{j\Omega_p} z^{-1}\right)^2} + \frac{a_2^*}{\left(1 - r_p e^{-j\Omega_p} z^{-1}\right)^2} + \ldots$$

Contributia lor in raspunsul la impuls este de forma:

$$\left[A_1 r_p^n \sin\left(\Omega_p n + \Phi_p\right) + A_2 n r_p^n \sin\left(\Omega_p n + \Phi_p\right)\right]\sigma[n].$$

Daca $r_p < 1$, raspunsul este amortizat-sistem stabil.

Daca $r_p > 1$, raspunsul creste in timp-sistem instabil.

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Calculul raspunsului unui SLIT discret caracterizat printr-o ecuatie cu diferente finite

Se considera SLIT cauzal, cu conditii initiale nenule (dar semnalul de la iesire este necauzal):

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k] \text{ , } a_0 \neq 0$$

$$\sum_{k=0}^{N} a_k Z_u\left\{y[n-k]\right\} = \sum_{k=0}^{M} b_k Z_u\left\{x[n-k]\right\}$$

$$\sum_{k=0}^{N} a_k z^{-k}\left[Y_u(z) + \sum_{n=1}^{k} y[-n] z^n\right] = \sum_{k=0}^{M} b_k z^{-k}\left[X_u(z) + \sum_{n=1}^{k} x[-n] z^n\right].$$

Daca semnalul de intrare este cauzal, $x[-n] = 0$ pentru $n > 0$:

$$\sum_{k=0}^{N} a_k z^{-k}\left[Y_u(z) + \sum_{n=1}^{k} y[-n] z^n\right] = \sum_{k=0}^{M} b_k z^{-k}\left[X_u(z)\right].$$

Cunoscand $X_u(z)$ si conditiile initiale, se determina $Y_u(z)$ si apoi $y[n]$. 42

Exemplu

$$y[n]-ay[n-1]=x[n],\ x[n]=ke^{j\Omega_0 n}\sigma[n],\ y[-1]\neq 0,$$

$$Y_u(z)-az^{-1}\left(Y_u(z)+y[-1]z\right)=\frac{k}{1-e^{j\Omega_0}z^{-1}};|z|>1.$$

$$Y_u(z)=\frac{k}{\left(1-e^{j\Omega_0}z^{-1}\right)\left(1-az^{-1}\right)}+\frac{ay[-1]}{1-az^{-1}}=$$

$$=\frac{-ke^{j\Omega_0}}{a-e^{j\Omega_0}}\frac{1}{1-e^{j\Omega_0}z^{-1}}+\frac{ka}{a-e^{j\Omega_0}}\frac{1}{1-az^{-1}}+\frac{ay[-1]}{1-az^{-1}}.$$

$$y[n]=\left(a^{n+1}y[-1]+\frac{ka^{n+1}}{a-e^{j\Omega_0}}+\frac{ke^{j\Omega_0(n+1)}}{a-e^{j\Omega_0}}\right)\sigma[n];|a|<1,|z|>1.$$

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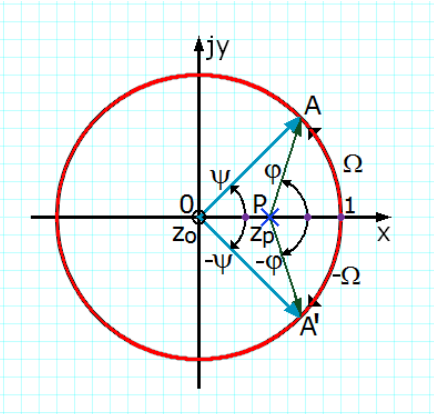
Sisteme de ordinul I

$$y[n]-ay[n-1]=kx[n]\Rightarrow H(z)=\frac{k}{1-az^{-1}}=\frac{kz}{z-a};z\in \mathrm{DC}.$$



$h[n]=ka^n\sigma[n].\ H(\Omega)=\frac{\left|\overline{OA}\right|}{\left|\overline{PA}\right|}e^{j(\psi-\varphi)}=\frac{1}{\left|\overline{PA}\right|}e^{j(\Omega-\varphi)}.$

Daca $0<a<1$

maximul raspunsului in frecventa

$\Rightarrow$ pentru $\left|\overline{PA}\right|$ minim, $\Omega=0$, filtru trece jos.

Daca $-1<a<0$,

maximul rezulta pentru $\Omega=\pi$, filtru trece sus.

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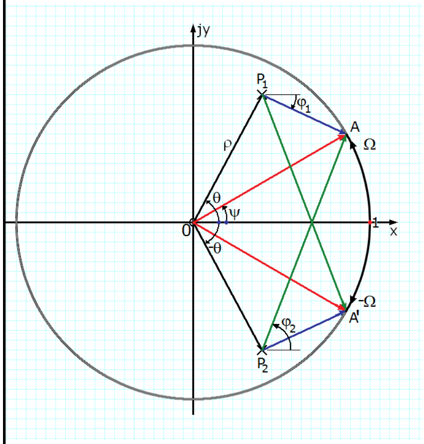
Sisteme de ordinul doi

$$y[n]+a_1y[n-1]+a_2y[n-2]=kx[n]\Rightarrow H(z)=\frac{k}{1+a_1z^{-1}+a_2z^{-2}}=\frac{kz^2}{z^2+a_1z+a_2}$$

Avem un zero de ordinul 2 in origine si polii $z_{p1,2}=\dfrac{-a_1+\sqrt{a_1^2-4a_2}}{2}$.



Daca $a_1^2<4a_2$ cei doi poli sunt complex conjugati $z_{p_{1,2}}=\rho e^{\pm j\theta}$.

Daca $a_1^2\geq 4a_2$ polii sunt reali.

Se considera sistemul cauzal si se cauta conditiile pe care trebuie sa le indeplinea sca coeficientii a_1 si a_2 pentru a obtine stabilitat ea.

$$\rho=\frac{\sqrt{a_1^2+4a_2-a_1^2}}{2}=\sqrt{a_2}<1;\,a_1^2<4a_2\Rightarrow$$

$\Rightarrow|a_2|<1,\, a_2>\dfrac{a_1^2}{4}$ sau daca polii sunt reali :

$-2<-a_1\pm\sqrt{a_1^2-4a_2}<2;\, a_1^2-4a_2\geq 0.$

Rezolvand sistemul de inegalitat i rezulta :

$a_1^2-4a_2\geq 0;\, a_2>-a_1-1;\, a_2>a_1-1.$

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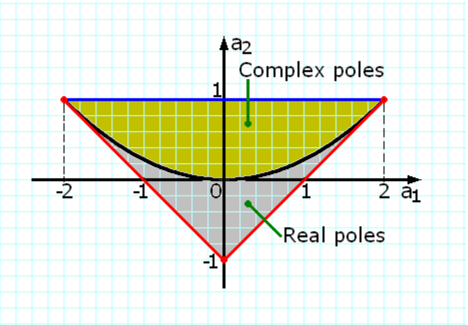

Parabola corespunde existentei unui singur pol real dublu.

Raspunsul in frecventa al sistemului : $H(\Omega)=k\dfrac{1}{\left|\overline{P_1A}\right|\left|\overline{P_2A}\right|}e^{j(2\Omega-\varphi_1-\varphi_2)}$.

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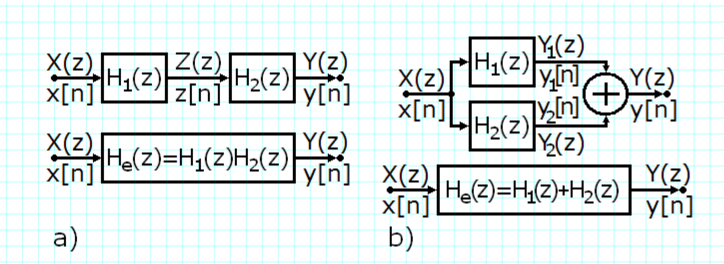
Functia de sistem echivalenta unor sisteme discrete conectate in serie si in paralel



$$h_e[n]=h_1[n]*h_2[n];$$
$$H_e(z)=H_1(z)H_2(z).$$

$$h_e[n]=h_1[n]+h_2[n];$$
$$H_e(z)=H_1(z)+H_2(z).$$

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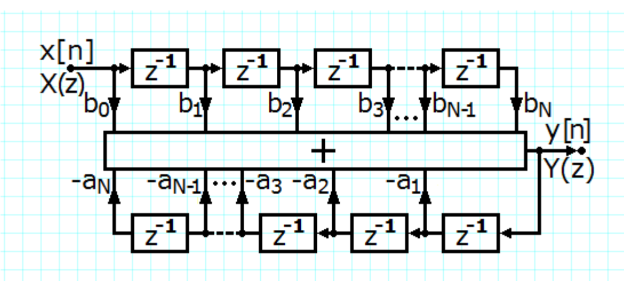
Forme de implementare ale filtrelor numerice utilizand transformarea Z

$$\sum_{k=0}^{N} a_k y[n-k]=\sum_{k=0}^{M} b_k x[n-k],\ a_0=1,\ M=N \Leftrightarrow y[n]=\sum_{k=0}^{M} b_k x[n-k]-\sum_{k=1}^{N} a_k y[n-k].$$



Forma specifica de reprezentare a implementarii in forma directa I. S-a considerat $a_0 = 1$

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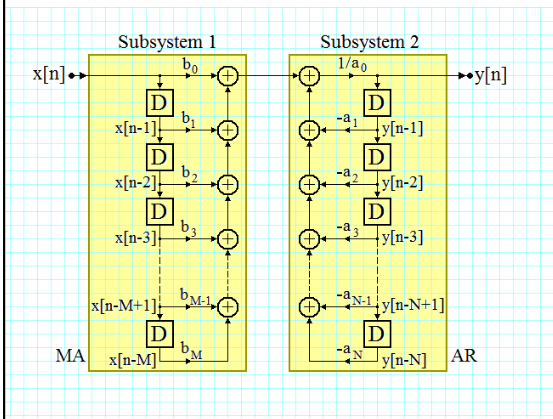
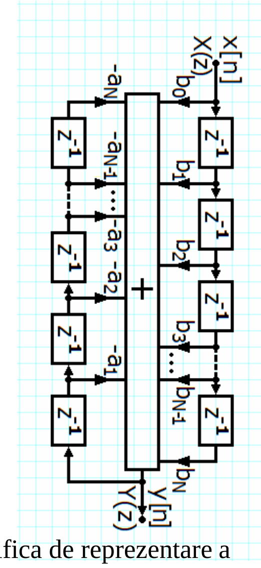


Forma directa I (N+M sumatoare, fiecare cu 2 intrari).



Forma specifica de reprezentare a implementarii in forma directa I. Un sumator cu 2N intrari. S-a considerat M=N, a_0=1.

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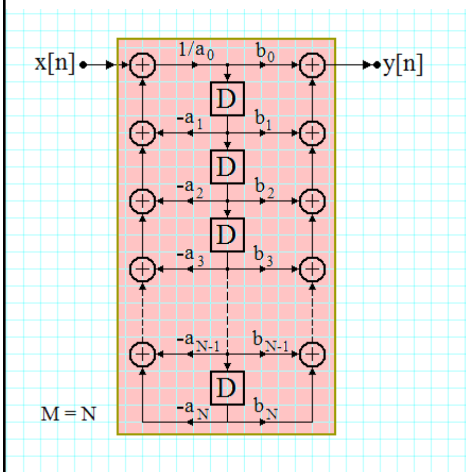
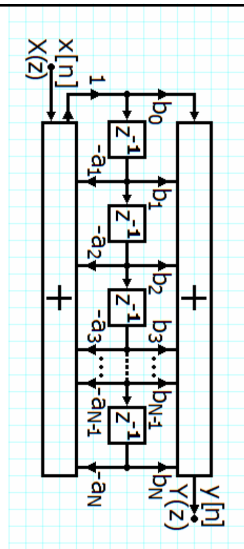


Forma directa II (2*N sumatoare, fiecare cu 2 intrari). S-a considerat M=N.



Forma specifica de reprezentare a implementarii in forma directa II. Doua sumatoare fiecare cu N intrari. S-a considerat M=N, a_0=1.

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