# Course 4: Baseband data transmission

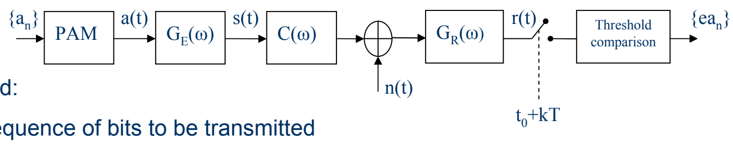
**“Data is a lot like humans: It is born. Matures. Gets married to other data, divorced. Gets old. One thing that it doesn't do is die. It has to be killed. ”**

Arthur Miller,
American playwright and essayist

# Agenda

- General model of the baseband data transmission systems
- ISI-less data transmission
- Nyquist filters: raised cosine/squared root raised cosine
- Baseband transmission with controlled ISI
- Evaluating the ISI level using the eye diagram
- Error probabilities for the baseband transmission

# Model for baseband data transmission




Legend:

- $a_n$: sequence of bits to be transmitted
- PAM: pulse amplitude modulator
- $G_E(\omega)$, $G_R(\omega)$: emission/reception filters (their transfer function)
- s(t): signal which carry data, transmitted in the channel
- $C(\omega)$: channel (physical environment) transfer function
- n(t): additive white noise
- r(t): received signal (based on it, a decision is made)
- Threshold comparison: e.g. positive value leads to “1”, negative to “0”
- $ea_n$: estimation of the received bits (ideally, identical to $a_n$)

# General remarks

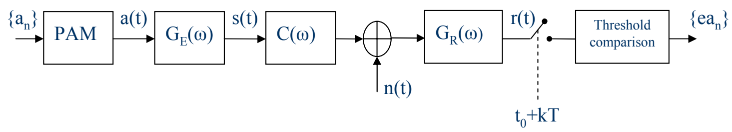
- For an ideal channel: $r(t) = As(t - \tau_0)$ (1)
- An “ideal channel” attenuates the input signal (A) and introduces a delay (tau)
- Both are constant !!! (they DO NOT DEPEND on the frequency)

$$C(\omega) = Ae^{-j\omega\tau_0} \quad (2)$$

- The derivate of the phase of C(ω) is called group delay, and is constant in this scenario
- In practice: $C(\omega) = A(\omega)e^{-j\varphi(\omega)}$ (3)
- The real channels distort the input signal and they are frequency-selective (dispersive)

Model for the baseband data transmission

## The baseband chain: mathematical approach [1]




- The PAM gives a first signal model for the data sequence:

$$a(t)=\sum_{n=0}^{N-1} a_n \delta(t-nT) \quad (4)$$

- $G_E$ "shapes" the signal for transmission:

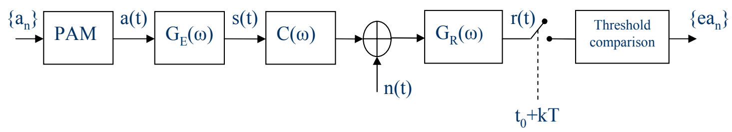
$$s(t)=a(t)*g_e(t)=\sum_{n=0}^{N-1} a_n g_e(t-nT) \quad (5)$$

$$g_e(t)=\frac{1}{2\pi}\int G_E(\omega)e^{j\omega t}d\omega \quad (6)$$

The pulse amplitude modulator (PAM) is an imaginary block; its output is a sequence of Dirac pulses, separated in time by the symbol time (T). Thus, for every bit of "1", a positive pulse is issued, whereas for every bit of "0" a negative pulse is generated. Although such pulses cannot be generated in practice, the PAM helps to the mathematical modeling of the transmission system.

$g_e(t)$ is a pulse shaping filter (at the transmission side). This filter shapes the signal, such a way that every digital information symbol is represented by a certain waveform g(t).

## The baseband chain: mathematical approach [2]




- If g(t) stands for the equivalent impulse response of the chain emission filter – channel – reception filter, then:

$$r(t) = \sum_{n=0}^{N-1} a_n g(t - nT) + n(t) \quad (7)$$

- The received signal is a weighted sum of g(t) waveforms, affected by additive noise

The received signal (eq. 7) is a weighted sum of g(t) shapes; g(t) is the equivalent impulse response of the assembly transmission filter – physical channel – reception filter. According to eq. 7, the received signal is a weighted sum of g(t) waveforms, perturbed by the additive noise n(t).

## The baseband chain: mathematical approach [3]

- In order to identify the k-th symbol, the corresponding k-th sample is collected by the receiver:

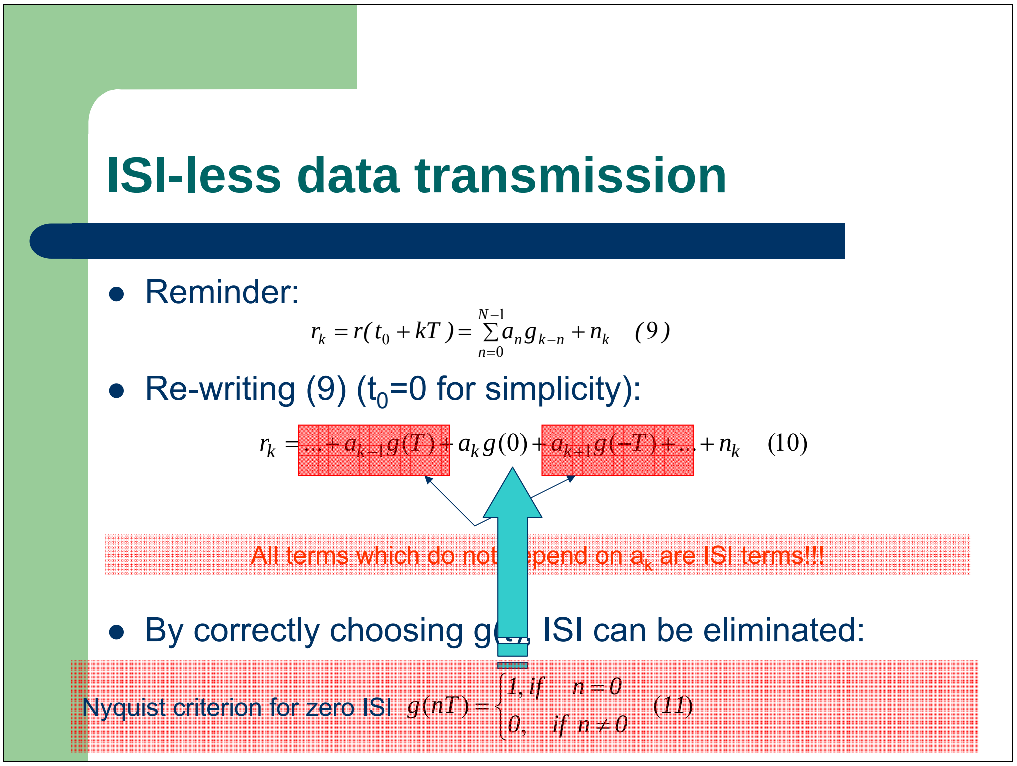
$$r_k = r(t_0 + kT) = \sum_{n=0}^{N-1} a_n g(t_0 + (k-n)T) + n(t_0 + kT) \quad (8)$$

- Re-written:

$$r_k = r(t_0 + kT) = \sum_{n=0}^{N-1} a_n g_{k-n} + n_k \quad (9)$$

- The k-th received sample depends on all the N transmitted symbols and on the sampled values of g(t) and n(t)

The received analog signal r(t) (but which conveys digital data) must be sampled, and a decision block will make an estimation of the transmitted bits. According to eq. (9), the kth received sample depend on all the transmitted bits $a_n$, although it would be desirable to depend only on the current (k-th) received sample.




The k-th received sample, $r_k$, is a weighted sum of the type $a_n g_{k-n}$, perturbed by the noise sample $n_k$ too. Ideally, $r_k$ should equal $a_k$, and in this case the received sample is identical to the transmitted symbol. Therefore, from eq. (10), only the middle term is useful, while all other terms are undesired. Thus, all terns depending on $a_n$ (excepting the middle one) introduce an influence of the adjacent transmitted symbols $a_n$ on the current bit: this influence is called ISI (Inter-Symbol Interference). Another adverse effect is the white noise, $n_k$.

ISI can be eliminated if the left and the right terms are “forced” to be 0 (Caution: the noise’s influence still exists!!!). This rule is met by Nyquist’s criterion for ISI-less data transmission (eq. 11).

Let’s “decode” now eq. (11): the transmission is free of ISI if the waveform g(t) exhibits regular zero crossing at all the sampling point from the reception side (nT), excepting for the current sample (n=0). Notice that the criterion only introduces a rule related to the sampling points: there is no constraint imposed on g(t), excepting those particular points. Theoretically, any waveform g(t) that meets this rule will generate no ISI, but this statement is true only if the receiver faithfully respect the ideal sampling points.

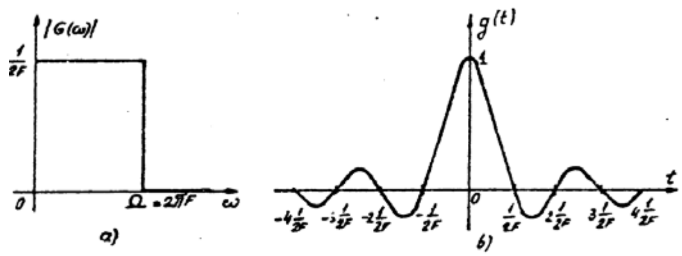
Reminder: $g(t)=g_e(t)*c(t)*g_r(t)$. Although the response c(t) only depends on the physical characteristics of the transmission environment, the transmitter and reception filter can be designed such a way that g(t) to meet Nyquist’s criterion.

ISI-less data transmission

# Nyquist theorem

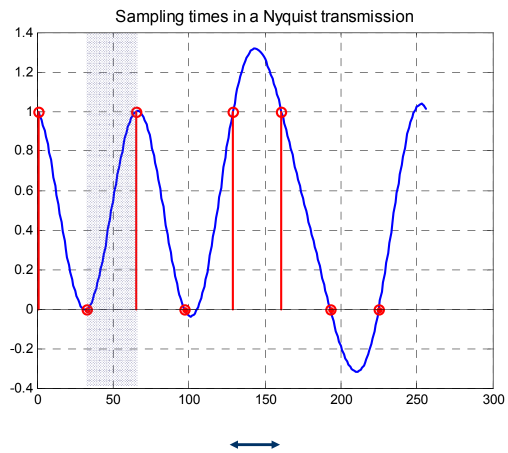
- In a channel which is equivalent with an ideal low-pass filter having the cutoff frequency F, it is possible to transmit symbols with a modulation rate equal or less to 2F symbols/sec, without ISI
- The characteristics of such a channel are shown below

Transfer function | Impulse response




According to the Nyquist theorem, the best spectral efficiency for an ISI-less data transmission (that is the highest ratio between R (rate) and W (bandwidth)) can be achieved if every symbol is shaped as a cardinal sine, which corresponds to an ideal LPF. In this case only, the transmission can be made at a symbol rate which is twice the bandwidth, while preserving it free of ISI. That's why this transmission rate is sometimes referred to as "ideal rate" or "Nyquist rate".

# Graphical view of the Nyquist theorem




Symbol time = $T_0/2=1/2F$

Symbol time = the time interval between the transmission in the channel of two consecutive information symbols.

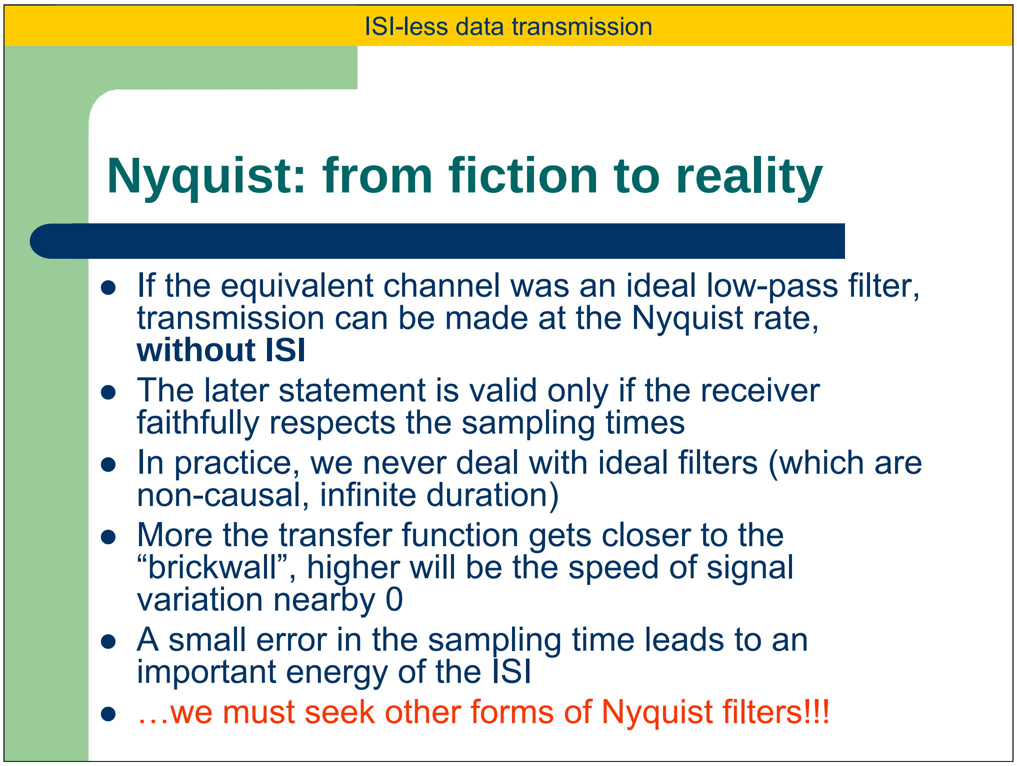
# Ideal case: a closer look

- The impulse response of the ideal low-pass filter is:

$$g(t) = \frac{\sin(\Omega t)}{\pi t} = \frac{\sin \frac{2\pi}{T_0} t}{\pi t} \quad (11)$$

- Such a waveform crosses zero every $T_0/2$ seconds
- In frequency, this corresponds to an ideal low-pass (brick-wall) filter, with the cut-off frequency $F=1/T_0$
- If a symbol is issued every $T_0/2$ seconds, transmission can be made without ISI
- The rate in this case will be 2F symbols/s (Nyquist rate, ideal rate)

Although, unlike in the digital baseband transmission (e.g.:NRZ), a single information symbol is represented by a waveform with infinite duration (a cardinal sine), we can transmit the next information symbol at the first zero crossing of the sinc waveform (i.e.: at $T_0/2$); this will prevent ISI to occur and will allow a transmission at a rate of $2/T_0$ symbols/sec. The later statement is identical with Nyquist's theorem, formulated on the previous slide.




The ideal LPF is unpractical, because its impulse response is infinite as duration and not-causal. In frequency, this characteristics are the basis for the ideal, rectangular ("brick-wall") form that corresponds to this filter.

Other disadvantages: the waveform evolution nearby the zero-crossing points is very "quick", so a small mistake in the sampling time will significantly impact the expected value of the sample. In practice, other forms of filters satisfying Nyquist criterion must be used.

# The road to the Nyquist filters [1]

- The equivalent response of the channel, at the sampling moments nT can be expressed as:

$$g_n = g(nT) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)e^{j\omega nT}d\omega = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\int_{(2k-1)\frac{\pi}{T}}^{(2k+1)\frac{\pi}{T}} G(\omega)e^{j\omega nT}d\omega \quad (12)$$

- By substituting $\omega \to \omega - \frac{2k\pi}{T}$ ,we get:

$$g_n = \frac{1}{2\pi}\int_{-\pi/T}^{\pi/T}\left[\sum_k G(\omega - \frac{2k\pi}{T})\right]e^{j\omega nT}d\omega \quad (13)$$

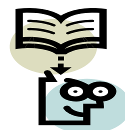
Starting with eq. (12), a mathematical method to find other filters that satisfy Nyquist criterion for zero ISI. Eq. (12), shows the expression of the n-th sample of g(t), expressed as the inverse Fourier transform of G(ω), computed at t=nT.

The term in the square brackets (rel. 13) is directly related to the spectrum of a shape that would satisfy ISI-less data transmission criterion.

# The road to the Nyquist filters [2]

- Reminder: the ideal LPF has the transfer function:

$$G(\omega)=\begin{cases}\frac{1}{2F}=T, & if\ |\omega|\le\frac{\pi}{T}\\ 0, & otherwise\end{cases}\quad (14)$$



T is the symbol time, which equals the zero-crossing rate of g(t), which is $T_0/2$ !!!

- By comparing (13) and (14), we obtain:

$$G_{eq}(\omega)=\begin{cases}\sum_k G(\omega-\frac{2k\pi}{T}), & if\ |\omega|\le\frac{\pi}{T}\\ 0, & if\ |\omega|>\frac{\pi}{T}\end{cases}\equiv\begin{cases}\frac{1}{2F}=T, & if\ |\omega|\le\frac{\pi}{T}\\ 0, & otherwise\end{cases}\quad (15)$$

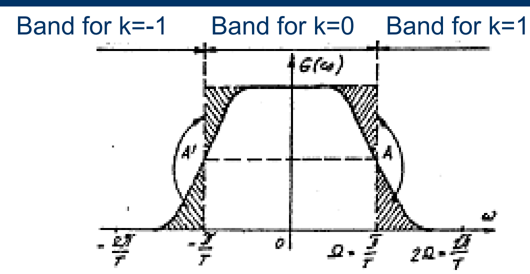
Any spectrum that satisfies eq. (15) corresponds to a waveform that meets Nyquist’s criterion. The spectra G(ω) that are inside the sum operator are not ideal LPF, but added up for k=-1,0 and 1 they will lead to an ideal LPF.

$$G_{eq}(\omega)=\begin{cases}\sum_k G(\omega-\frac{2k\pi}{T}), & if\ |\omega|\le\frac{\pi}{T}\\ 0, & if\ |\omega|>\frac{\pi}{T}\end{cases}\equiv\begin{cases}\frac{1}{2F}=T, & if\ |\omega|\le\frac{\pi}{T}\\ 0, & otherwise\end{cases}\quad (15)$$

## The ultimate consequence of (15)…

- Ideal LPF from (14) can be approximated "piece-by-piece" ("k by k" from 15)
- Any transfer function G which respects (15) can be used as Nyquist filter

# The road to the Nyquist filters [3]




- G(ω) satisfies Nyquist's zero ISI criterion if it exhibits symmetry around π/T (A and A') points
- In this case, the side bands (k=-1 and k=1) "compensate" the frequency response of the main lobe (k=0), such a way that eq. (15) is satisfied
- A common choice for G is provided by the raised cosine filters

It is straightforward that, in order to satisfy (15), the spectrum G(ω) must exhibit symmetry around the cut-off frequency Ω=π/T, as shown in the figure above. If G(ω) and the ideal LPF determine equal areas to the left and to the right of Ω, then G(ω) corresponds to a waveform that meets Nyquist criterion.

# The raised-cosine family

- The previously formulated criteria is met if:

$$G(\omega)=\begin{cases}\dfrac{\pi}{\Omega}=T, & |\omega|\le\Omega(1-\alpha)\\ \dfrac{\pi}{2\Omega}\left[1-\sin\left(\dfrac{\pi\omega}{2\alpha\Omega}-\dfrac{\pi}{2\alpha}\right)\right], & \Omega(1-\alpha)\le|\omega|\le\Omega(1+\alpha)\\ 0, & |\omega|>\Omega(1+\alpha)\end{cases}\qquad(16)$$

- The impulse response will be:

$$g(t)=\frac{\sin(\Omega t)}{\Omega t}\cdot\frac{\cos(\alpha\Omega t)}{1-\left(\dfrac{2\alpha\Omega t}{\pi}\right)^2}\qquad(17)$$

There is a whole category of filters that meet Nyquist’s criterion and respect the symmetry of their spectra around the cut-off frequency. These filters are referred to as Nyquist pulse-shaping filters, or “raised-cosine” filters, because the shape of their spectrum, which looks pretty much like a cosine in the frequency domain. Both their frequency (16) and their impulse response (17) depend on a parameter α, called *roll-off factor* or *excess bandwidth*, which spans from 0 to 1.

## Comments on the choice of the roll-off factor

- $\alpha$ spans from 0 to 1 and is referred to as excess bandwidth or roll-off factor
- For $\alpha=0$ the ideal low-pass filter is obtained, whereas $\alpha=1$ defines the square cosine filter
- Common choices in practice range from 0.1 to 0.5
- If we want to achieve a data rate of R,, then the bandwidth we need will be:

$$B = (1+\alpha)\frac{R}{2} \quad (18)$$

What lies behind the "excess bandwidth" name?

The answer is given by relation (18). Basically, "excess bandwidth" means the extra bandwidth we need, when compared to the bandwidth we would need to achieve a fixed data rate of R when using cardinal sine shapes for the data pulses.

Or, otherwise formulated: the data rate R can be reached with a consumption of bandwidth of B=R/2 when the pulses are cardinal sine pulses, and with a consumption of bandwidth B=(1+α)(R/2) when we use other types of Nyquist waveforms than the ideal cardinal sine.

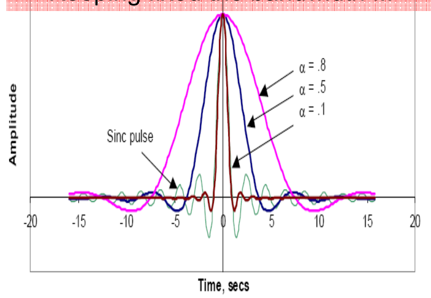
# The need for Nyquist filters

- These filters are usually referred to as “pulse-shaping” filters
- A certain shape of the data pulses will lead to a certain level of spectral efficiency
- Interpreting Nyquist: “Take care how you shape the data pulses, and you will achieve the efficiency and the robustness you desire”
- Keeping α low, good spectral efficiency is obtained (☺), but high sensitivity to the sampling time accuracy too (☹)
- In noisy channels square root raised cosine filters must be used to the transmitter and to the receiver

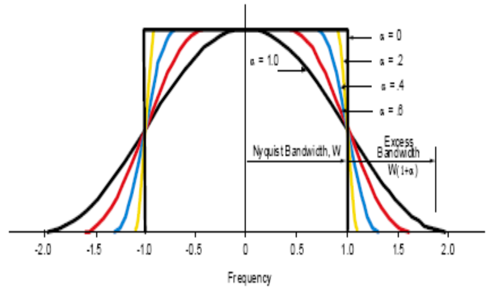
ISI-less data transmission

## Nyquist filters: when Time meets Frequency

Keeping fixed the bandwidth !!!




Keeping fixed the data rate !!!



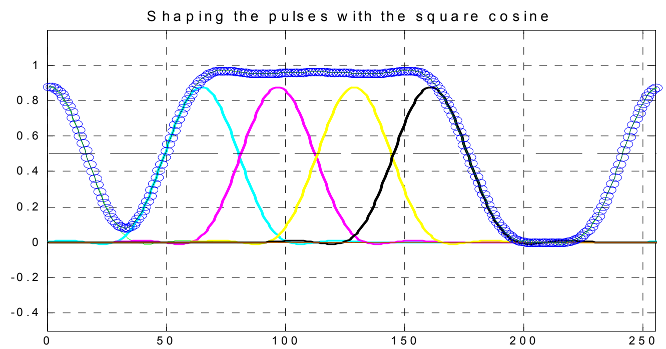

- Small roll-off factor means higher transmission rate at the same occupied bandwidth
- …BUT! The energy of the side-lobes in time is higher, which causes higher sensitivity in case of synchronization problems

*Pictures downloaded from www.complextoreal.com

The above figures highlight the need for a good compromise when choosing α. Thus, the left figure shows various impulse responses of some raised cosine filters. A small value for α will permit a higher data rate (the zero crossings of the waveforms appear at smaller time intervals); nevertheless, the side-lobes carry, in this case, an important amount of energy. The sensitivity to the synchronization errors will be higher than for high values of α (close to 1), when the side-lobes have smaller amplitudes.

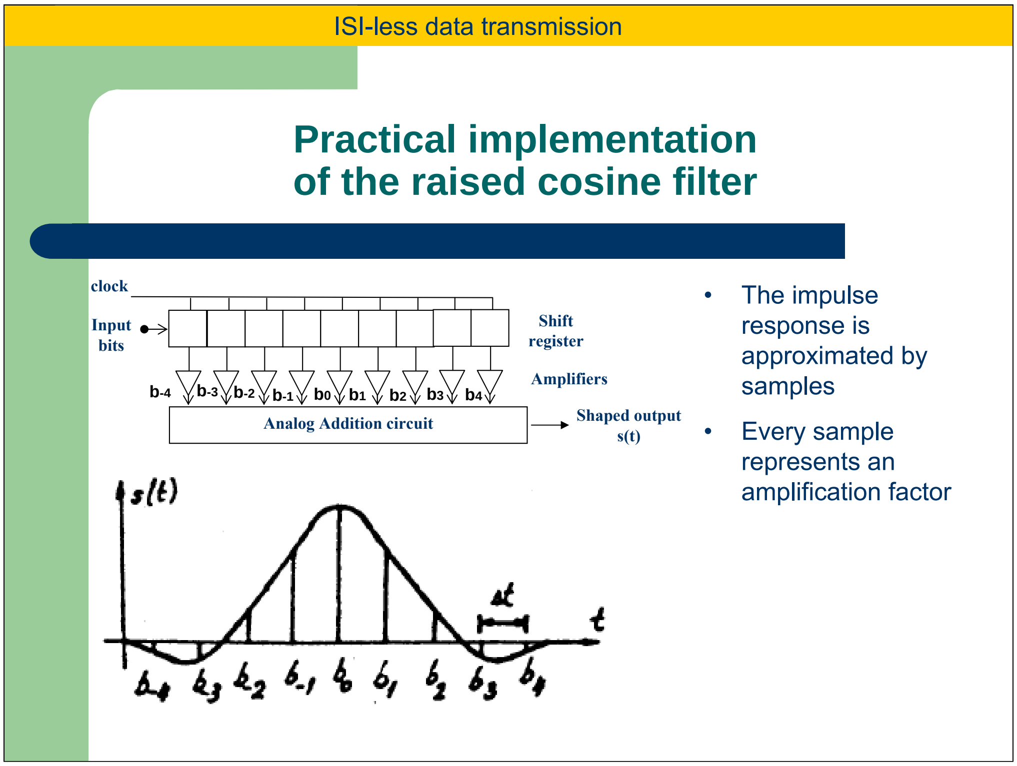
The right figure shows the frequency responses of several filters from the raised cosine family; when plotting these responses, we consider a fixed transmission rate. The lowest bandwidth consumption, as expected, is given by the brick-wall and higher will be the value of α, higher the extra bandwidth required, compared to the brick-wall.

# Squared cosine case




- Very low-energy sidelobes
- Symbol time doubled compared to the sinc case

In the case of α=1, we get the squared cosine filter. As shown in the above figure, its side-lobes are almost negligible, which is a good property from the ISI robustness point of view. The dashed line has the meaning of a comparison threshold, always needed in practice, in order to make accurate decisions on the transmitted symbols. E.g.: sample>0.5 means “1”, sample<0.5 means “0”.




Although the raised cosine filters, as defined by equations (16, 17) are closer to the practical implementation than the ideal LPF, they still have some properties that makes their exact implementation impossible: their impulse response is infinite, and they are not causal. In practice, we need to apply a time window of finite duration on the impulse response, and to shift this windowed version to the right, such as to obtain a causal system. The two effects are shown on the above figure, where the impulse response has finite duration and is 0 for negative values on the time axis.

In practice, these filters are implemented using digital circuits. That is, instead of having an analog waveform like in the figure above, we will get some samples of this waveform. The most common implementation relies on FIR (Finite Impulse Response) filters, that have a finite number of coefficients. Their values are given by the amplitude of the samples taken from the original waveform. Such a filter can be implemented using a transversal structure, as shown in the upper figure from this slide. This implementation uses a shift register with N cells (N=9 in our case). The result is that we get a FIR filter with 9 taps. Most of the digital transmission devices (and especially in wireless transmission) incorporate such a filter.

## Transmission with controlled ISI

- Motivation:
  - In practice, the raised cosine filters will not reach the Nyquist rate
  - Closer they are to this objective, higher is the transmission sensitivity to synchronization errors
- Solution:
  - Some degree of ISI can be tolerated, if the ISI is "controlled"
  - Higher rates can be obtained
  - The cosine filter allows to reach the Nyquist rate
- Principle: every waveform g(t) will carry two-bits of information
- This is the reason for calling this transmission "duo-binary"

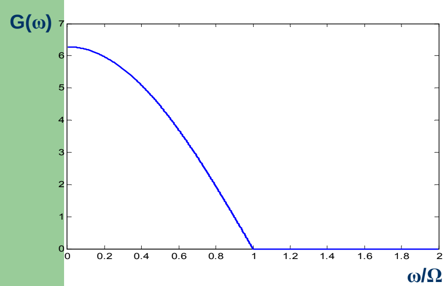
The raised cosine filters can be we well approximated in practice, but they can only achieve $1/1+\alpha$ of the Nyquist rate. If we want a transmission robust to ISI, high values should be chosen for $\alpha$. For example, if $\alpha=0.5$, only 66% of the Nyquist rate can be obtained. This drawback is the main motivation behind the transmission with controlled ISI. This transmission, commonly referred to as duo binary transmission, sacrifices the "ISI-less" principle, but allows to reach the Nyquist rate. ISI is controlled, in the sense that, by system's design, the ISI influence is accurately known and the phenomenon can be counteracted at receiver side.

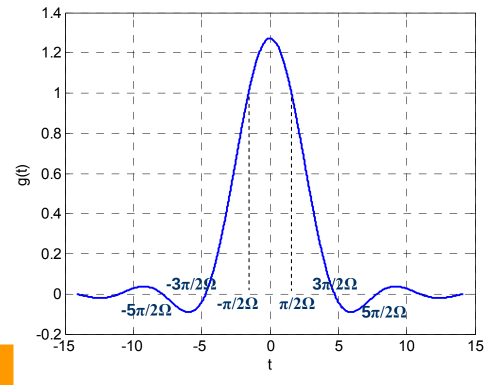
Transmission with controlled ISI

# The cosine filter

$$G(\omega)=\begin{cases}\dfrac{2\pi}{\Omega}\cos\dfrac{\pi\omega}{2\Omega} & |\omega|\le\Omega\\ 0, \quad if\ |\omega|>\Omega\end{cases} \qquad (19)$$

$$g(t)=\frac{4}{\pi}\frac{\cos(\Omega t)}{1-\left(\dfrac{2\Omega t}{\pi}\right)^2} \qquad (20)$$

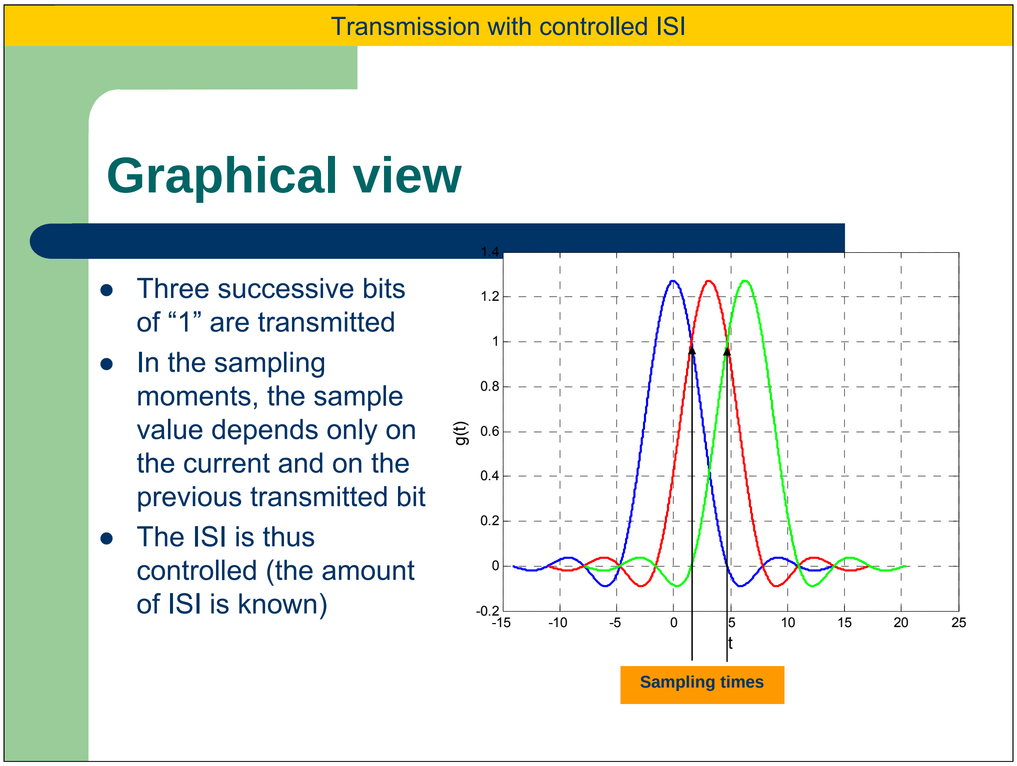



$$T=\frac{\pi}{\Omega}=\frac{1}{2F} \qquad (21)$$

The signaling rate is 2F!!!

When, instead of a raised cosine, a cosine filter (eq. 19,20) is used for shaping purposes, the signal so generated will lead to a transmission with controlled ISI.




It can be seen from the figure above how, at the sampling instants the collected samples depend not on a single transmitted symbol, but on exactly two transmitted symbols (the current one and the previous one). That’s why the transmission is called with controlled ISI (ISI exists, but it’s known and, therefore, it can be controlled/counteracted).

Transmission with controlled ISI

# Mathematical approach

- The k-th sample can be expressed as:

$$r_k = r(-\frac{T}{2}+kT) = a_k g(-\frac{T}{2}) + a_{k-1} g(\frac{T}{2}) + \sum_{n \neq k,k-1} a_n g[(k-n-\frac{1}{2})T] \quad (24)$$

- In the right-side sum, all the terms are zero (they match exactly the zero-crossing of g). It follows that:

$$r_k = a_k + a_{k-1} \quad (25)$$

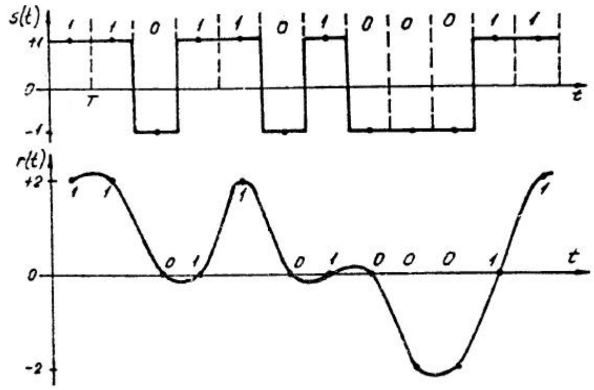
- Theoretically, if $a_k$ is bi-polarly encoded $r_k$ may take three values:

$$r_k = \begin{cases} 0, & \text{if } a_k \neq a_{k-1} \\ 2, & \text{if } a_k = a_{k-1} = 1 \\ -2, & \text{if } a_k = a_{k-1} = -1 \end{cases} \quad (26)$$

The k-th received sample can be described according to (24). If cosine waveforms are used to shape the bits and the sampling times are faithfully respected, the right side sum is zero, and $r_k$ will only depend on $a_k$ and on $a_{k-1}$.

Transmission with controlled ISI

## Graphical view




A signal with controlled ISI is show in the figure above. Thus, two successive bits of “1” will issue a collected sample of amplitude 2; when a “1” is followed by a “0” ore vice-versa, the sample will be 0; finally, two bits of “0” will issue a sample of value -2.

The main disadvantage of a transmission with controlled ISI is the fact that the decision on the current bit, e.g. $a_k$, depends on the decision of the previous bit, $a_{k-1}$. E.g.: if $r_k$=0, we have to know the value of the previous bit, in order to make a decision about the current one. If $a_{k-1}$ was, for example, detected as a “0”, it follows that $a_k$ is “1”.

If the $a_{k-1}$ is wrongly detected, the error propagates until a new error will turn the decision to the right path.

Transmission with controlled ISI

## Pre-coding

- Disadvantage of controlled ISI: every decision depends on two successive bits (error propagation)
- Solution: pre-coding performed
- Instead of $a_k$, another sequence ($b_k$) is transmitted, computed as:

$$b_k = a_k \oplus b_{k-1} \qquad (27)$$

- This leads to a one-sample based decision, as follows:

$$a_k = b_k \oplus b_{k-1} = \begin{cases} 0, & if \ r_k = \pm 2 \\ 1, & if \ r_k = 0 \end{cases} \qquad (28)$$

The pre-coding, at the transmitter side, removes the dependency of the decision on the two bits. Somehow, this inter-bit dependency is “transferred” at the transmitter side. This can be simply done by means of a pre-coding operation.

The transmission steps are now:

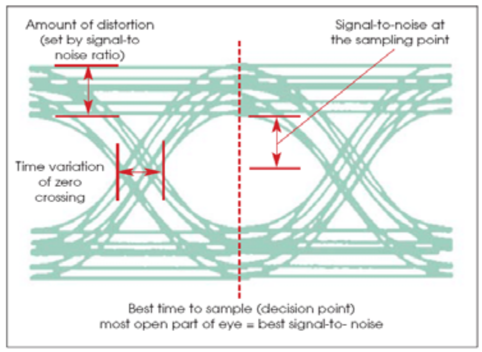
1. The original bits $a_n$ are randomly generated
2. The precoded bits $b_n$ are computed
3. The precoded bits are bipolarly encoded (i.e.1 -> +1V, 0 -> -1V).
4. The signal so obtained is passed through a cosine filter, to generate a transmission with controlled ISI.

Thus, instead of transmitting directly the “original” bits $a_n$, some new bits $b_n$ (the “pre-coded” stream) are generated, according to (27). Taking into account this pre-coding, the receiver has to inverter, in order to extract $a_n$, the original bits, inversion shown by equation (28). The reception steps are:

1. The signal is sampled ($r_k$) values are collected.
2. Decision is made:
   - if $r_k$ is 2 or -2, it follows that $b_k = b_{k-1}$, and $a_k = 0$ (see eq. 27)
   - if $r_k$ is 0, it follows that $b_k \neq b_{k-1}$ and $a_k = 1$

## Eye diagram…at a glance

- The eye diagram allows to evaluate the degree of ISI
- It is obtained by superposing the time domain representation of the signal for a large number of transmitted symbols

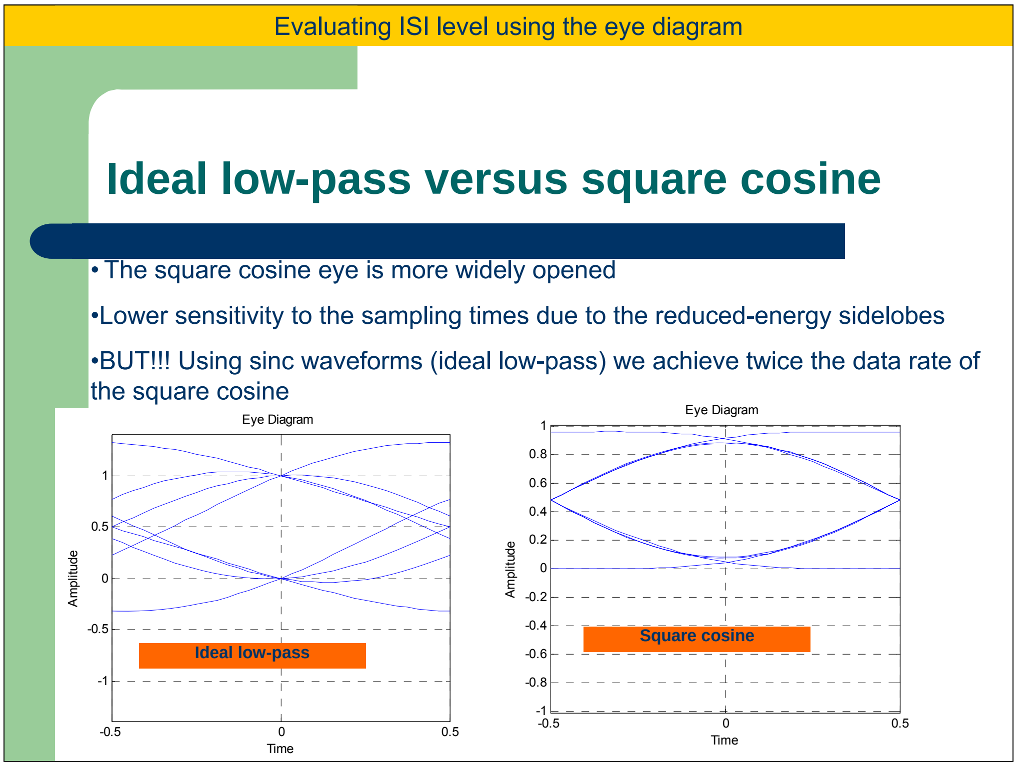



*Picture downloaded from the paper “Analyzing signals using eye diagram”, author G. Breed, High Frequency Electronics Journal

The eye diagram is a very useful tool to assess the accuracy of a digital transmission. It can be applied to rectangular or raised cosine waveforms (those are the two practical cases that can be met in a digital transmission).

The eye diagram is an oscilloscope representation on a time domain signal, in which waveforms for every T sec. interval are superposed. E.g. Waveform that represent a data signal in the interval [0,T] is superposed with the waveform for the interval [T,2T], with the waveform for the interval [2T,3T] and so on.

In a radio system, the point of measurement for the eye diagram may be prior to the modulator in a transmitter, or following the demodulator in a receiver, depending on which portion of the system requires examination. The eye diagram can also be used to examine signal integrity in a purely digital base-band system—such as fiber optic transmission, network cables or on a circuit board. The figure above shows the type of information that is given by the eye diagram.

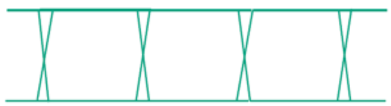



In the slide, we compare the eye diagram from two pulse-shaping waveforms (the two extreme cases of raised cosine).

For the cardinal sine (roll-off α=0), the eye is not so widely open, pointing out high sensitivity to the sampling point accuracy. By the contrary, in the case of square cosine (roll-off α=0), the eye is widely open, the transmission is not sensitive to the synchronization errors, but barely one half of the Nyquist rate can be achieved.

Evaluating ISI level using the eye diagram

## Eye diagram for unfiltered rectangular waveforms



- Ideal eye diagram of a square waveform (no jitter, no noise)

(a)

- Misalignment of rise and fall times (jitter)

(b)

- At higher rates, jitter has more impact (even if its absolute values decreases)

Figures above show some eye diagrams for purely digital systems, in which the data signal is made of nearly rectangular waveforms. The jitter error is clearly highlighted in figures 2 and 3, where misalignment of rise and fall times causes a kind of “grid”, or spreading, to occur towards the end of the bit intervals.

# Improving the performance of the baseband transmission systems

- The the k-th received symbol may be expressed as:

$$r_k = r(t_0 + kt) = \sum_{n=-N}^{N} a_n g_{k-n} + n_k \quad (29)$$

- The error probability for the k=0-th transmitted symbol is:

$$P_e = P\left\{ \left| \sum_{\substack{n=-N \\ n\neq 0}}^{N} a_n g_{-n} + n_0 \right| > dg_0 \right\} \quad (30)$$

- Assuming that there is no useful signal in the channel (only AWGN noise with unitary variance), the noise power after the receiver filter is:

$$\sigma_{fn}^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| G_R(\omega) \right|^2 d\omega \quad (31)$$

*the fn subscript above emphasizes that we refer to the power of the filtered noise

## SNR maximization

- If we assume that there is no noise in the channel, the 0-th received sample can be computed as:

$$r_0 = r(t)\big|_{t=0} = \frac{1}{2\pi}\int_{-\infty}^{\infty} G_R(\omega)S(\omega)e^{j\omega t}d\omega\big|_{t=0} = \frac{1}{2\pi}\int_{-\infty}^{\infty} G_R(\omega)S(\omega)d\omega \quad (32)$$

- At t=0, we try to choose $G_R(\omega)$ which maximizes $\frac{|r_0|^2}{\sigma_{fn}^2}$
- Solution: $G_R(\omega)=S^*(\omega)$ (an expression of the matched filter)
- For an ideal channel, $C(\omega)=1$ and the received signal form is determined by the emitter's shaping filter: $S(\omega)=G_E(\omega)$
- The choice which maximizes the SNR: $G_E(\omega) = G_R(\omega) = \sqrt{G(\omega)}$
- Example: for the square root square cosine filter, we have:

$$G_E(\omega) = G_R(\omega) = \cos\frac{\pi\omega}{4\Omega} \quad (33)$$

In the digital transmission systems we deal with two disturbing phenomena: noise and ISI. While the later can be counteracted using Nyquist filter, we need to focus our attention on the first effect too. There is a huge amount of detection/estimation literature, proving that if the noise is of AWGN type, the best solution is to place a matched filter at the receiver side.

A well balanced choice, which counteracts both effects is to use a pair of filters (square root raised cosine), one of which is the pulse shaping filter (at transmitter) and the other one that is a non optimal matched filter (at the receiver). Together, they will lead to an equivalent response that respects Nyquist's criterion for zero ISI, while the receiver's filter will effectively remove the noise.

## The performance of the baseband transmission systems

- Error probability
- Transmission speed
- Transmission efficiency

## The error probability in ideal channels [1]

- An ideal channel for data transmission does not modify the signal transmitted through it (C(ω)=1)
- Identical transmitter and receiver filters may be used
- The average power of the transmitted signal is:

$$P_S = \frac{\left\langle a^2 \right\rangle}{2\pi} \int_{-\infty}^{\infty} \left| \sqrt{G(\omega)} \right|^2 d\omega = \left\langle a^2 \right\rangle \quad (34)$$

$\left\langle a^2 \right\rangle$ is the average power of the transmitted symbol

## The error probability in ideal channels [2]

- We assume M-level signaling, with the levels: $\pm d, \pm 3d, \pm 5d, \ldots$
- The useful energy is conserved by the shaping filter:

$$\langle a^2 \rangle = \frac{d^2}{3}(M^2 - 1) \quad (35)$$

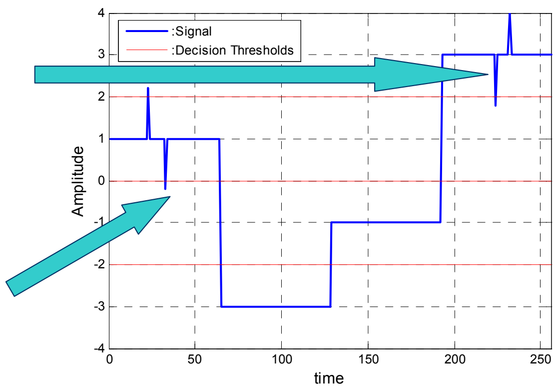
- Decision thresholds at: $0, \pm 2d, \pm 4d, \ldots$
- The error probability is:

$$P_e = (1 - \frac{1}{M}) P\left( |n_0| > d \right) \quad (36)$$

- ISI is considered 0 in equation (36)

# Discussion on equation 36

- Reminder: $P_e = (1-\frac{1}{M})P\left(|n_0| > d\right) \quad (36)$
- For the two extreme levels, an error occurs only if noise sample has a certain sign
- For the other levels, the sign doesn't meter



## The error probability in ideal channels [3]

- Taking into account the pdf of the noise in (36):

$$P_e = (1-\frac{1}{M})\frac{1}{\sqrt{2\pi}\sigma}\int_d^{\infty} e^{-\frac{n^2}{2\sigma^2}}dn \quad (37)$$

- erfc is introduced by:

$$erfc(x) = \frac{2}{\sqrt{\pi}}\int_x^{\infty}\exp(-z^2)dz \quad (38)$$

- Error probability can be express as (from eq. 37,38)

$$P_e = \frac{1}{2}(1-\frac{1}{M})erfc\left(\frac{d}{\sigma\sqrt{2}}\right) \quad (39)$$

# The error probability in ideal channels [4]

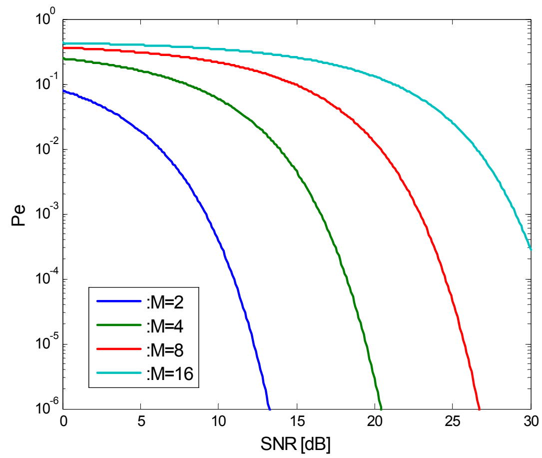
- From eq. (35), d can be expressed as:

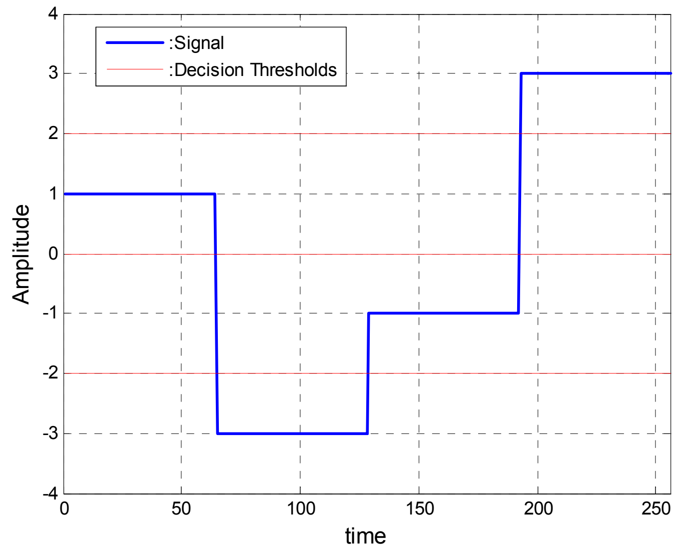
$$d = \sqrt{\frac{3}{M^2 - 1} \cdot P_S} \quad (40)$$

- From (39) and (40):

$$P_e = \frac{1}{2}(1 - \frac{1}{M}) erfc\left( \sqrt{\frac{3}{2(M^2 - 1)}} \cdot \sqrt{\frac{P_S}{P_N}} \right) \quad (41)$$

# The error probability in ideal channels [5]





:Signal
:Decision Thresholds
Amplitude
time
4
3
2
1
0
-1
-2
-3
-4
0
50
100
150
200
250

The probability that the noise sample value to exceed the threshold (d=2 in our example) is the delimited area under the Gaussian curve!!!